ABSTRACT. A classical theorem of Szegö states that for functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ convex in $|z| < 1$, the sequence of partial sums $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ must be convex in $|z| < \frac{1}{4}$. For the more general family consisting of functions of the form $z + \sum_{k=2}^{\infty} a_k z^{n_k}$, where $\{n_k\}$ denotes an increasing (finite or infinite) sequence of integers ($\geq 2$), we find the radius of convexity ($\approx 0.21$) and the radius of starlikeness ($\approx 0.37$). The extremal function in both cases is $z + z/(1-z) = z + \sum_{k=2}^{\infty} z^k$.

INTRODUCTION

Denote by $S_0$ the family consisting of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic and starlike univalent in $\Delta = \{z||z|<1\}$ and by $K$ the subfamily of $S_0$ containing the convex functions. It is an old and well-known result of Szegö [9] that for any $N \geq 2$,

$$f_N(z) = z + \sum_{n=2}^{N} \frac{a_n}{4^{n-1}} z^n \in K \quad \text{if} \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K,$$

in other words, any section of a function in $K$ is convex univalent for $|z| < \frac{1}{4}$. Let $\{n_k\}_{k=2}^{\infty}$ denote an arbitrary increasing sequence (finite or infinite) of positive integers with $k \leq n_k$; in a recent paper [8] one of us conjectured that if $f \in K$ as above, then

$$z + \sum_{k=2}^{\infty} \frac{a_{n_k}}{4^{n_k-1}} z^{n_k} \in K \quad \text{and} \quad z + \sum_{k=2}^{\infty} \frac{a_{n_k}}{2^{n_k-1}} z^{n_k} \in S_0.$$

This would generalize Szegö's theorem to more general sequences.

To see that this conjecture is false, we consider the function $g(z) = z + \sum_{k=1}^{\infty} z^{2k} = z + z^2/(1-z^2)$ associated with the convex function $f(z) = z + \sum_{k=2}^{\infty} z^k = z/(1-z)$. Then $1 + zg''(z)/g'(z) = 0$ at $z = -c \sim -0.20936$ and $\frac{1}{4}$. Let $\{n_k\}_{k=2}^{\infty}$ denote an arbitrary increasing sequence (finite or infinite) of positive integers with $k \leq n_k$; in a recent paper [8] one of us conjectured that if $f \in K$ as above, then

$$z + \sum_{k=2}^{\infty} \frac{a_{n_k}}{4^{n_k-1}} z^{n_k} \in K \quad \text{and} \quad z + \sum_{k=2}^{\infty} \frac{a_{n_k}}{2^{n_k-1}} z^{n_k} \in S_0.$$
$z g'(z)/g(z) = 0$ at $z = -b \sim -0.3715$. It is the object of this note to show that the above $g$ is extremal for the radii of convexity and starlikeness. In fact we shall prove

**Theorem 1.** If $\{n_k\}_{k=2}^{\infty}$ is an increasing sequence of integers with $k \leq n_k$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$, then $\hat{f}(z) = z + \sum_{k=2}^{\infty} a_{n_k} z^{n_k}$ is convex univalent in $|z| < c$ where $c \sim 0.20936$ is the unique root in $(0,1)$ of the equation

$$x \frac{1 + x^2}{(1 - x^2)^3} = \frac{1}{4}.$$  

**Theorem 2.** If $\{n_k\}_{k=2}^{\infty}$ is an increasing sequence of integers with $k \leq n_k$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$ then $\hat{f}(z) = z + \sum_{k=2}^{\infty} a_{n_k} z^{n_k}$ is starlike univalent in $|z| < b \sim 0.3715$ where $b$ is the unique root in $(0,1)$ of the equation

$$x \frac{1}{(1 - x^2)^2} = \frac{1}{2}.$$  

Since $zf' \in S_0$ if and only if $f \in K$, a consequence of Theorem 1 follows.

**Corollary.** Using the notation of Theorem 1, if $f \in S_0$ then $\hat{f}$ is starlike univalent in $|z| < c$.

There exists considerable literature concerning sections of univalent functions and we refer the reader to [2, Chapter 8] for a general survey. In particular it has been established (see [2, 4, 5] or [8]) that in the proofs of Theorem 1 and Theorem 2 we may assume $f(z) = z/(1 - z)$. This is due to a convolution theorem of Ruscheweyh and Sheil–Small [7].

Let the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic in $\Delta$; their convolution (or Hadamard product) is the function $f \ast g$ defined by

$$f \ast g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$  

For each $T \in \mathbb{R}$ we define

$$h_T(z) \equiv \frac{z/(1 - z)^2 + iTz/(1 - z)}{1 + iT}.$$  

Clearly a locally univalent function $u(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is convex in $\Delta$ if and only if $(zu'(z) \ast h_T(z))/z \neq 0$ for all real $T$ and $z \in \Delta$. Our proofs rely on the following results, due essentially to Ruscheweyh (see [3, 6]).

**Theorem A.** Let $u(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$ such that

$$\inf_{T \in \mathbb{R}} \left| \frac{zu'(z) \ast h_T(z)}{z} \right| \geq \delta > 0.$$  

Then $v(z) = z + \sum_{n=2}^{\infty} b_n z^n \in K$, provided that $\sum_{n=2}^{\infty} n^2 |a_n - b_n| \leq \delta$. 

Note that Theorem A is an extension of a classical result [1]: the case $u(z) \equiv z$ is usually called Alexander's Theorem.

**Theorem B.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_0$ such that

$$\inf_{z \in \Delta} \left| \frac{f \ast h_T(z)}{z} \right| \geq \delta > 0.$$ 

Then $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S_0$ provided that $\sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta$.

Note [6] that the conclusion of Theorem B holds for any $f \in K \subset S_0$ with $\delta = \frac{1}{4}$.

**Proof of Theorem 1**

As noticed above it is sufficient to prove Theorem 1 for $f(z) = z/(1 - z)$. For sequences $\{n_k\}_{k=2}^{\infty}$ satisfying $n_2 = 2$ and $n_3 > 3$ our approach is to prove that $v(z) = z + \sum_{k=2}^{\infty} c^{n_k-1} Z^{n_k} \in K$ by showing that it either satisfies the condition of Alexander's Theorem or the more general Theorem A with $u(z) = z/(1 - cz)$.

As already mentioned, if $\{n_k\}_{k=2}^{\infty} = \{2k - 2\}_{k=2}^{\infty}$ then we obtain the extremal function $z + \sum_{k=2}^{\infty} z^{2(k-1)} = z + z^2/(1 - z^2)$.

For the remaining sequences $\{n_k\}_{k=2}^{\infty}$ we now show that $z + \sum_{k=2}^{\infty} z^{n_k}$ is convex univalent in $|z| < \frac{1}{4}$ by using Theorem A with an appropriate choice of the function $u(z)$. If $n_2 > 2$, then

$$z + \sum_{k=2}^{\infty} \frac{1}{4^{n_k-1}} z^{n_k} \in K$$

because

$$\sum_{k=2}^{\infty} \frac{n_k^2}{4^{n_k-1}} \leq \sum_{j=3}^{\infty} \frac{j^2}{4^{j-1}} = \frac{104}{108} < 1.$$ 

Now consider the function $u(z) = z/(1 - z/4)$. A simple calculation gives, for all real $T$ and $z \in \Delta$,

$$\left| \frac{zu'(z) \ast h_T(z)}{z} \right| = |h_T(z/4)| \geq \frac{1 - |z|/4}{(1 + |z|/4)^3} > \frac{48}{125} = .384.$$ 

Since

$$u(z) \equiv z + \frac{1}{4} z^2 + \frac{1}{4^2} z^3 + \frac{1}{4^3} z^4 + \sum_{j=5}^{\infty} \frac{1}{4^{j-1}} z^j,$$

we now obtain from Theorem A

$$v(z) \equiv z + \frac{1}{4} z^2 + \frac{1}{4^2} z^3 + \frac{1}{4^3} z^4 + \sum_{k=5}^{\infty} \frac{1}{4^{n_k-1}} z^{n_k} \in K$$

because, since $n_5 \geq 5$,

$$\sum_{n=2}^{\infty} n^2 \left| \frac{u^{(n)}(0)}{n!} - \frac{v^{(n)}(0)}{n!} \right| \leq \sum_{j=5}^{\infty} \frac{j^2}{4^{j-1}} < .151 < .384.$$
In a similar way if
\[ u(z) = \frac{z}{1-z/4} - \frac{1}{43}z^4, \]
we compute
\[ \left| \frac{zu'(z) * h_T(z)}{z} \right| = \left| h'_T(z/4) - \frac{1}{42} \frac{4 + iT}{1 + iT} z^3 \right| \geq .384 - .250 = .134 \]
and, therefore,
\[ v(z) = z + \frac{1}{4} z^2 + \frac{1}{4} z^3 + \frac{1}{4} z^5 + \sum_{k=6}^{\infty} \frac{1}{4^{n_k-1}} z^{n_k} \in K, \]
\[ k \geq 4 \]

since
\[ \sum_{n=2}^{\infty} n^2 \left| \frac{u^{(n)}(0)}{n!} - \frac{v^{(n)}(0)}{n!} \right| \leq \sum_{j=6}^{\infty} \frac{j^2}{4^{j-1}} < .053 < .134. \]

Finally let \( u(z) = z + \frac{1}{4} z^2 + \frac{1}{16} z^3 \). Because, for any \( z \in \Delta \),
\[ \inf_T \left| \frac{zu'(z) * h_T(z)}{z} \right| = \frac{|2 + \frac{3}{2} z + \frac{3}{4} z^2| - |\frac{1}{2} z + \frac{3}{8} z^2|}{2} \]
we have
\[ \left| \frac{zu'(z) * h_T(z)}{z} \right| \geq \frac{|2 + \frac{3}{2} z + \frac{3}{4} z^2|^2 - |\frac{1}{2} z + \frac{3}{8} z^2|^2}{2(2 + \frac{3}{2} + \frac{3}{4} + \frac{1}{2} + \frac{3}{8})} \].

Setting \( z = e^{i\theta} \), we see that the numerator of this last expression is \( 219/64 + (63/8) \cos \theta + 6 \cos^2 \theta \), which attains its minimum when \( \cos \theta = -21/32 \). Thus,
\[ \left| \frac{zu'(z) * h_T(z)}{z} \right| \geq \frac{429/512}{41/4} > .08 > .053. \]

We get from Theorem A and (4),
\[ v(z) = z + \frac{1}{4} z^2 + \frac{1}{16} z^3 + \sum_{k=6}^{\infty} \frac{1}{4^{n_k-1}} z^{n_k} \in K, \]
\[ k \geq 4 \]

Now it should be clear (see (1), (2), (3), (5)) that the theorem is valid (even with the constant \( c \) replaced by \( \frac{1}{4} \)) for any sequence \( \{n_k\}_{k=2}^{\infty} \) except possibly for some sequences with \( n_2 = 2 \) and \( n_3 \geq 4 \). The rest of the proof consists of proving \( z + cz^2 + \sum_{n_k \geq 4} c^{n_k-1} z^{n_k} \in K \).
We need the following inequalities:

\[ (6) \quad \sum_{n=1}^{m} (2n)^2 c^{2n-1} + \sum_{n=2m+3}^{n=\infty} n^2 c^{n-1} \leq 1, \quad m \geq 1. \]

\[ (7) \quad \sum_{n=2}^{m} (2n-1)^2 c^{2n-2} + \sum_{n=2m+2}^{n=\infty} n^2 c^{n-1} \leq \frac{1-c}{(1+c)^3}, \quad m \geq 2. \]

Since \( \sum_{n=1}^{\infty} (2n)^2 c^{2n-1} = 1 \), the inequality (6) will follow if we can show that the left-hand side is an increasing function of \( m \). This is equivalent to

\[ \sum_{n=1}^{m} (2n)^2 c^{2n-1} + \sum_{n=2m+3}^{n=\infty} n^2 c^{n-1} \leq \sum_{n=1}^{m+1} (2n)^2 c^{2n-1} + \sum_{n=2m+5}^{n=\infty} n^2 c^{n-1} \]

or

\[ (2m+3)^2 c + (2m+4)^2 c^2 \leq (2m+2)^2, \quad m \geq 1. \]

Now the result follows because

\[ 1 - \left( \frac{2m+3}{2m+2} \right)^2 c - \left( \frac{2m+4}{2m+2} \right)^2 c^2 \]

is minimized when \( m = 1 \) and

\[ 1 - \left( \frac{5}{4} \right)^2 c - \left( \frac{7}{4} \right)^2 c^2 > 0. \]

It is easily established that \( \sum_{n=2}^{\infty} (2n-1)^2 c^{2n-2} = (1-c)/(1+c)^3 \) and (7) will follow if we can show that the left-hand side is an increasing function of \( m \). This amounts to

\[ (2m+2)^2 c + (2m+3)^2 c^2 \leq (2m+1)^2, \quad m \geq 2. \]

This last statement is valid because

\[ 1 - \left( \frac{2m+2}{2m+1} \right)^2 c - \left( \frac{2m+3}{2m+1} \right)^2 c^2 \]

is minimized when \( m = 2 \) and

\[ 1 - \left( \frac{6}{5} \right)^2 c - \left( \frac{9}{5} \right)^2 c^2 > 0. \]

Clearly we obtain by (6) and Alexander's theorem, for all \( m \geq 1 \),

\[ f(z) \equiv z + \sum_{n=1}^{m} c^{2n-1} z^{2n} + \sum_{n_k \geq 2m+3} c^{n_k-1} z^{n_k} \in K. \]

Now let us consider

\[ u(z) = \frac{z}{1-cz} \]

\[ = z + \sum_{n=1}^{m} c^{2n-1} z^{2n} + c^2 m z^{2m+1} + \sum_{n=2}^{m} c^{2n-2} z^{2n-1} + \sum_{j=2m+2}^{\infty} c^{j-1} z^j. \]
We compute for all \( z \in \Delta \) and \( T \in \mathbb{R} \),

\[
\left| \frac{z \beta'(z) * h_T(z)}{z} \right| = |h_T'(cz)| \geq \frac{1 - c|z|}{(1 + c|z|)^3} > \frac{1 - c}{(1 + c)^3}.
\]

It then follows from (7), (9), and Theorem A that

\[
v(z) = z + \sum_{n=1}^{m} c^{2n-1} z^{2n} + c^{2m} z^{2m+1} + \sum_{n_k \geq 2m+2} c^{n_k-1} z^{n_k} \in K
\]

for all \( m \geq 1 \).

We now can complete our proof of Theorem 1. As noticed before we need to prove that

\[
f(z) = z + z^2 + \sum_{n_k \geq 4} z^{n_k}
\]

is convex univalent in \( |z| < c \). Consider an increasing sequence of integers \( \{n_k\} \) where \( n_1 \geq 4 \) and define \( 2j + 1 \geq 5 \) as the smallest odd number in this sequence (if \( 2j + 1 \) does not exist, our counterexample to the conjecture shows clearly that \( f \) is convex univalent in \( |z| < c \)). Then either

\[
f(z) = z + z^2 + \sum_{k=1}^{N} z^{2j_k} + z^{2j+1} + \sum_{n_k \geq 2j+2} z^{n_k}
\]

for some sequence \( 2 \leq t_1 < t_2 \cdots < t_N \leq j \), \( 1 \leq N \leq j - 1 \), or

\[
f(z) = z + z^2 + z^{2j+1} + \sum_{n_k \geq 2j+2} z^{n_k}.
\]

In the second case \( f \) is convex in \( |z| < c \), by (8) with \( m = 1 \). In the first case a finite number of applications of (8) with \( m = 2, \ldots \) show that we can assume \( t_1 = 2, t_2 = 3, \ldots, t_N = j \), that is we need to prove

\[
f(z) = z + \sum_{n=1}^{j} z^{2n} + z^{2j+1} + \sum_{n_k \geq 2j+2} z^{n_k}
\]

to be convex in \( |z| < c \). But this follows from (10) with \( m = j \). Our proof of Theorem 1 is now complete.

**Proof of Theorem 2**

It suffices to show that \( g(z) = z + \sum_{k=2}^{\infty} b^{n_k-1} z^{n_k} \in S_0 \). If \( n_2 \geq 3 \), then \( \sum_{k=2}^{\infty} n_k b^{n_k-1} \leq \sum_{n=3}^{\infty} nb^{n-1} < 1 \), a well-known sufficient condition for \( g \) to be in \( S_0 \). If \( n_2 = 2, n_3 = 3, \) and \( n_4 = 4 \), we apply Theorem B with \( f(z) = z/(1 - bz) \in K \) and \( \delta = \frac{1}{4} \). If \( n_2 = 2, n_3 = 3, \) and \( n_4 > 4 \), we can apply
Theorem B with \( f(z) = z/(1 - bz) - b^3z^4 \) and \( \delta = \frac{1}{4} \) because

\[
\left| \frac{f * h_T(z)}{z} \right| = \frac{h_T(bz)}{bz} - \frac{b^3(4 + iT)z^3}{1 + iT} \\
\geq \left| \frac{h_T(bz)}{bz} \right| - 4b^3|z|^3 \\
> \frac{1}{(1 + b)^2} - 4b^3 > \frac{1}{4}.
\]

In both cases we obtain \( g \in S_0 \) because

\[
\sum_{n=2}^{\infty} n \left| \frac{f^{(n)}(0)}{n!} - \frac{g^{(n)}(0)}{n!} \right| \leq \sum_{n=5}^{\infty} nb^{n-1} < .17 < 1/4.
\]

Thus Theorem 2 is valid except possibly when \( n_2 = 2 \) and \( n_3 \geq 4 \).

The remainder of the proof follows along the lines of that of Theorem 1, with appropriate modifications that include substituting \( b \) for \( c \), Theorem B for Theorem A, and replacing (6) and (7) with

\[
(6') \quad \sum_{n=1}^{m} 2nb^{2n-1} + \sum_{n=2m+3}^{\infty} nb^{n-1} \leq 1, \quad m \geq 1.
\]

\[
(7') \quad \sum_{n=2}^{m} (2n - 1)b^{2n-2} + \sum_{n=2m+2}^{\infty} nb^{n-1} \leq \frac{1}{(1+b)^2}, \quad m \geq 2.
\]

References


