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RADII PROBLEMS FOR GENERALIZED SECTIONS OF CONVEX FUNCTIONS

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ABSTRACT. A classical theorem of Szëgo states that for functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ convex in $|z| < 1$, the sequence of partial sums $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ must be convex in $|z| < \frac{1}{4}$. For the more general family consisting of functions of the form $z + \sum_{k=2}^{\infty} a_{n_k} z^{n_k}$, where $\{n_k\}$ denotes an increasing (finite or infinite) sequence of integers (≥ 2), we find the radius of convexity (≈ 0.21) and the radius of starlikeness (≈ 0.37). The extremal function in both cases is $z + z^2/(1 - z^2) = z + \sum_{k=1}^{\infty} z^{2k}$ associated with the convex function $z/(1 - z) = z + \sum_{k=2}^{\infty} z^k$.

INTRODUCTION

Denote by S_0 the family consisting of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic and starlike univalent in $\Delta = \{z \mid |z| < 1\}$ and by K the subfamily of S_0 containing the convex functions. It is an old and well-known result of Szegö [9] that for any $N \geq 2$,

$$f_N(z) = z + \sum_{n=2}^{N} \frac{a_n}{4^{n-1}} z^n \in K \quad \text{if } f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K,$$

in other words, any section of a function in K is convex univalent for $|z| < \frac{1}{4}$. Let $\{n_k\}_{k=2}^{\infty}$ denote an arbitrary increasing sequence (finite or infinite) of positive integers with $k \leq n_k$; in a recent paper [8] one of us conjectured that if $f \in K$ as above, then

$$z + \sum_{k=2}^{\infty} \frac{a_{n_k}}{4^{n_k - 1}} z^{n_k} \in K \quad \text{and} \quad z + \sum_{k=2}^{\infty} \frac{a_{n_k}}{2^{n_k - 1}} z^{n_k} \in S_0.$$

This would generalize Szegö's theorem to more general sequences.

To see that this conjecture is false, we consider the function $g(z) = z + \sum_{k=1}^{\infty} z^{2k} = z + z^2/(1 - z^2)$ associated with the convex function $f(z) = z + \sum_{k=2}^{\infty} z^k = z/(1 - z)$. Then $1 + zg''(z)/g'(z) = 0$ at $z = -c \sim -.20936$ and

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$zg'(z)/g(z) = 0$ at $z = -b \sim -.3715$. It is the object of this note to show that the above g is extremal for the radii of convexity and starlikeness. In fact we shall prove

Theorem 1. *If $\{n_k\}_{k=2}^{\infty}$ is an increasing sequence of integers with $k \le n_k$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$, then $\tilde{f}(z) = z + \sum_{k=2}^{\infty} a_{n_k} z^{n_k}$ is convex univalent in $|z| < c$ where $c \sim .20936$ is the unique root in $(0,1)$ of the equation*

$$x\frac{1+x^2}{(1-x^2)^3} = \frac{1}{4}.$$

Theorem 2. *If $\{n_k\}_{k=2}^{\infty}$ is an increasing sequence of integers with $k \le n_k$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$ then $\tilde{f}(z) = z + \sum_{k=2}^{\infty} a_{n_k} z^{n_k}$ is starlike univalent in $|z| < b \sim .3715$ where b is the unique root in $(0,1)$ of the equation*

$$\frac{x}{(1-x^2)^2} = \frac{1}{2}.$$

Since $zf' \in S_0$ if and only if $f \in K$, a consequence of Theorem 1 follows.

Corollary. *Using the notation of Theorem 1, if $f \in S_0$ then $\tilde{f}$ is starlike univalent in $|z| < c$.*

There exists considerable literature concerning sections of univalent functions and we refer the reader to [2, Chapter 8] for a general survey. In particular it has been established (see [2, 4, 5] or [8]) that in the proofs of Theorem 1 and Theorem 2 we may assume $f(z) \equiv z/(1-z)$. This is due to a convolution theorem of Ruscheweyh and Sheil–Small [7].

Let the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic in Δ; their convolution (or Hadamard product) is the function $f * g$ defined by

$$f * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For each $T \in \mathbf{R}$ we define

$$h_T(z) \equiv \frac{z/(1-z)^2 + iTz/(1-z)}{1+iT}.$$

Clearly a locally univalent function $u(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is convex in Δ if and only if $(zu'(z) * h_T(z))/z \neq 0$ for all real T and $z \in \Delta$. Our proofs rely on the following results, due essentially to Ruscheweyh (see [3, 6]).

Theorem A. *Let $u(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$ such that*

$$\inf_{\substack{T \in \mathbf{R} \\ z \in \Delta}} \left| \frac{zu'(z) * h_T(z)}{z} \right| \ge \delta > 0.$$

Then $v(z) = z + \sum_{n=2}^{\infty} b_n z^n \in K$, provided that $\sum_{n=2}^{\infty} n^2 |a_n - b_n| \le \delta$.

Note that Theorem A is an extension of a classical result [1]: the case $u(z) \equiv z$ is usually called Alexander's Theorem.

Theorem B. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_0$ *such that*

$$\inf_{\substack{t \in \mathbf{R} \\ z \in \Delta}} \left| \frac{f * h_T(z)}{z} \right| \geq \delta > 0 .$$

Then $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S_0$ *provided that* $\sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta$.

Note [6] that the conclusion of Theorem B holds for any $f \in K \subset S_0$ with $\delta = \frac{1}{4}$.

Proof of Theorem 1

As noticed above it is sufficient to prove Theorem 1 for $f(z) = z/(1-z)$. For sequences $\{n_k\}_{k=2}^{\infty}$ satisfying $n_2 = 2$ and $n_3 > 3$ our approach is to prove that $v(z) = z + \sum_{k=2}^{\infty} c^{n_k - 1} z^{n_k} \in K$ by showing that it either satisfies the condition of Alexander's Theorem or the more general Theorem A with $u(z) = z/(1 - cz)$. As already mentioned, if $\{n_k\}_{k=2}^{\infty} = \{2k-2\}_{k=2}^{\infty}$ then we obtain the extremal function $z + \sum_{k=2}^{\infty} z^{2(k-1)} = z + z^2/(1 - z^2)$.

For the remaining sequences $\{n_k\}_{k=2}^{\infty}$ we now show that $z + \sum_{k=2}^{\infty} z^{n_k}$ is convex univalent in $|z| < \frac{1}{4}$ by using Theorem A with an appropriate choice of the function $u(z)$. If $n_2 > 2$, then

$$z + \sum_{k=2}^{\infty} \frac{1}{4^{n_k - 1}} z^{n_k} \in K \tag{1}$$

because

$$\sum_{k=2}^{\infty} \frac{n_k^2}{4^{n_k - 1}} \leq \sum_{j=3}^{\infty} \frac{j^2}{4^{j-1}} = \frac{104}{108} < 1 .$$

Now consider the function $u(z) = z/(1 - z/4)$. A simple calculation gives, for all real T and $z \in \Delta$,

$$\left| \frac{zu'(z) * h_T(z)}{z} \right| = |h_T'(z/4)| \geq \frac{1 - |z|/4}{(1 + |z|/4)^3} > \frac{48}{125} = .384 .$$

Since

$$u(z) \equiv z + \frac{1}{4} z^2 + \frac{1}{4^2} z^3 + \frac{1}{4^3} z^4 + \sum_{j=5}^{\infty} \frac{1}{4^{j-1}} z^j ,$$

we now obtain from Theorem A

$$v(z) \equiv z + \frac{1}{4} z^2 + \frac{1}{4^2} z^3 + \frac{1}{4^3} z^4 + \sum_{k=5}^{\infty} \frac{1}{4^{n_k - 1}} z^{n_k} \in K \tag{2}$$

because, since $n_5 \geq 5$,

$$\sum_{n=2}^{\infty} n^2 \left| \frac{u^{(n)}(0)}{n!} - \frac{v^{(n)}(0)}{n!} \right| \leq \sum_{j=5}^{\infty} \frac{j^2}{4^{j-1}} < .151 < .384 .$$

In a similar way if

$$u(z) \equiv \frac{z}{1 - z/4} - \frac{1}{4^3} z^4,$$

we compute

$$\left| \frac{zu'(z) * h_T(z)}{z} \right| = \left| h_T'(z/4) - \frac{1}{4^2} \frac{4 + iT}{1 + iT} z^3 \right|$$
$$\geq .384 - .250 = .134$$

and, therefore,

$$v(z) \equiv z + \frac{1}{4} z^2 + \frac{1}{4^2} z^3 + \frac{1}{4^4} z^5 + \sum_{k=6}^{\infty} \frac{1}{4^{n_k - 1}} z^{n_k} \in K, \tag{3}$$

since

$$\sum_{n=2}^{\infty} n^2 \left| \frac{u^{(n)}(0)}{n!} - \frac{v^{(n)}(0)}{n!} \right| \leq \sum_{j=6}^{\infty} \frac{j^2}{4^{j-1}} < .053 < .134. \tag{4}$$

Finally let $u(z) \equiv z + \frac{1}{4} z^2 + \frac{1}{16} z^3$. Because, for any $z \in \Delta$,

$$\inf_T \left| \frac{zu'(z) * h_T(z)}{z} \right| = \frac{|2 + \frac{3}{2} z + \frac{3}{4} z^2| - |\frac{1}{2} z + \frac{3}{8} z^2|}{2}$$

we have

$$\left| \frac{zu'(z) * h_T(z)}{z} \right| \geq \frac{|2 + \frac{3}{2} z + \frac{3}{4} z^2|^2 - |\frac{1}{2} z + \frac{3}{8} z^2|^2}{2(2 + \frac{3}{2} + \frac{3}{4} + \frac{1}{2} + \frac{3}{8})}.$$

Setting $z = e^{i\theta}$, we see that the numerator of this last expression is $219/64 + (63/8)\cos\theta + 6\cos^2\theta$, which attains its minimum when $\cos\theta = -21/32$. Thus,

$$\left| \frac{zu'(z) * h_T(z)}{z} \right| \geq \frac{429/512}{41/4} > .08 > .053.$$

We get from Theorem A and (4),

$$v(z) \equiv z + \frac{1}{4} z^2 + \frac{1}{16} z^3 + \sum_{k=6}^{\infty} \frac{1}{4^{n_k - 1}} z^{n_k} \in K. \tag{5}$$

Now it should be clear (see (1), (2), (3), (5)) that the theorem is valid (even with the constant c replaced by $\frac{1}{4}$) for any sequence $\{n_k\}_{k=2}^{\infty}$ except possibly for some sequences with $n_2 = 2$ and $n_3 \geq 4$. The rest of the proof consists of proving $z + cz^2 + \sum_{n_k \geq 4} c^{n_k - 1} z^{n_k} \in K$.

We need the following inequalities:

$$\sum_{n=1}^{m}(2n)^2c^{2n-1}+\sum_{n=2m+3}^{\infty}n^2c^{n-1}\le 1,\qquad m\ge 1. \tag{6}$$

$$\sum_{n=2}^{m}(2n-1)^2c^{2n-2}+\sum_{n=2m+2}^{\infty}n^2c^{n-1}\le \frac{1-c}{(1+c)^3},\qquad m\ge 2. \tag{7}$$

Since $\sum_{n=1}^{\infty}(2n)^2c^{2n-1}=1$, the inequality (6) will follow if we can show that the left-hand side is an increasing function of m. This is equivalent to

$$\sum_{n=1}^{m}(2n)^2c^{2n-1}+\sum_{n=2m+3}^{\infty}n^2c^{n-1}\le\sum_{n=1}^{m+1}(2n)^2c^{2n-1}+\sum_{n=2m+5}^{\infty}n^2c^{n-1}$$

or

$$(2m+3)^2c+(2m+4)^2c^2\le(2m+2)^2,\qquad m\ge 1.$$

Now the result follows because

$$1-\left(\frac{2m+3}{2m+2}\right)^2c-\left(\frac{2m+4}{2m+2}\right)^2c^2$$

is minimized when $m=1$ and

$$1-(\tfrac{5}{4})^2c-(\tfrac{3}{2})^2c^2>0.$$

It is easily established that $\sum_{n=2}^{\infty}(2n-1)^2c^{2n-2}=(1-c)/(1+c)^3$ and (7) will follow if we can show that the left-hand side is an increasing function of m. This amounts to

$$(2m+2)^2c+(2m+3)^2c^2\le(2m+1)^2,\qquad m\ge 2.$$

This last statement is valid because

$$1-\left(\frac{2m+2}{2m+1}\right)^2c-\left(\frac{2m+3}{2m+1}\right)c^2$$

is minimized when $m=2$ and

$$1-(\tfrac{6}{5})^2c-(\tfrac{7}{5})^2c^2>0.$$

Clearly we obtain by (6) and Alexander's theorem, for all $m\ge 1$,

$$f(z)\equiv z+\sum_{n=1}^{m}c^{2n-1}z^{2n}+\sum_{n_k\ge 2m+3}c^{n_k-1}z^{n_k}\in K. \tag{8}$$

Now let us consider

$$\begin{aligned}u(z)&=\frac{z}{1-cz}\\&=z+\sum_{n=1}^{m}c^{2n-1}z^{2n}+c^{2m}z^{2m+1}+\sum_{n=2}^{m}c^{2n-2}z^{2n-1}+\sum_{j=2m+2}^{\infty}c^{j-1}z^{j}.\end{aligned}$$

We compute for all $z \in \Delta$ and $T \in \mathbf{R}$,

$$\left|\frac{zu'(z) * h_T(z)}{z}\right| = |h_T'(cz)| \geq \frac{1 - c|z|}{(1 + c|z|)^3} > \frac{1 - c}{(1 + c)^3}. \tag{9}$$

It then follows from (7), (9), and Theorem A that

$$v(z) = z + \sum_{n=1}^{m} c^{2n-1} z^{2n} + c^{2m} z^{2m+1} + \sum_{n_k \geq 2m+2} c^{n_k - 1} z^{n_k} \in K \tag{10}$$

for all $m \geq 1$.

We now can complete our proof of Theorem 1. As noticed before we need to prove that

$$f(z) = z + z^2 + \sum_{n_k \geq 4} z^{n_k}$$

is convex univalent in $|z| < c$. Consider an increasing sequence of integers $\{n_k\}$ where $n_1 \geq 4$ and define $2j + 1 \geq 5$ as the smallest odd number in this sequence (if $2j + 1$ does not exist, our counterexample to the conjecture shows clearly that f is convex univalent in $|z| < c$). Then either

$$f(z) = z + z^2 + \sum_{k=1}^{N} z^{2t_k} + z^{2j+1} + \sum_{n_k \geq 2j+2} z^{n_k}$$

for some sequence $2 \leq t_1 < t_2 \cdots < t_N \leq j$, $1 \leq N \leq j - 1$, or

$$f(z) = z + z^2 + z^{2j+1} + \sum_{n_k \geq 2j+2} z^{n_k}.$$

In the second case f is convex in $|z| < c$, by (8) with $m = 1$. In the first case a finite number of applications of (8) with $m = 2, \ldots$ show that we can assume $t_1 = 2$, $t_2 = 3, \ldots, t_N = j$, that is we need to prove

$$f(z) = z + \sum_{n=1}^{j} z^{2n} + z^{2j+1} + \sum_{n_k \geq 2j+2} z^{n_k}$$

to be convex in $|z| < c$. But this follows from (10) with $m = j$. Our proof of Theorem 1 is now complete.

Proof of Theorem 2

It suffices to show that $g(z) = z + \sum_{k=2}^{\infty} b^{n_k - 1} z^{n_k} \in S_0$. If $n_2 \geq 3$, then $\sum_{k=2}^{\infty} n_k b^{n_k - 1} \leq \sum_{n=3}^{\infty} n b^{n-1} < 1$, a well-known sufficient condition for g to be in S_0. If $n_2 = 2$, $n_3 = 3$, and $n_4 = 4$, we apply Theorem B with $f(z) = z/(1 - bz) \in K$ and $\delta = \frac{1}{4}$. If $n_2 = 2$, $n_3 = 3$, and $n_4 > 4$, we can apply

Theorem B with $f(z) = z/(1-bz) - b^3z^4$ and $\delta = \frac{1}{4}$ because

$$\begin{aligned}\left|\frac{f * h_T(z)}{z}\right| &= \left|\frac{h_T(bz)}{bz} - \frac{b^3(4+iT)}{1+iT}z^3\right| \\ &\geq \left|\frac{h_T(bz)}{bz}\right| - 4b^3|z|^3 \\ &> \frac{1}{(1+b)^2} - 4b^3 > \frac{1}{4}.\end{aligned}$$

In both cases we obtain $g \in S_0$ because

$$\sum_{n=2}^{\infty} n\left|\frac{f^{(n)}(0)}{n!} - \frac{g^{(n)}(0)}{n!}\right| \leq \sum_{n=5}^{\infty} nb^{n-1} < .17 < 1/4.$$

Thus Theorem 2 is valid except possibly when $n_2 = 2$ and $n_3 \geq 4$.

The remainder of the proof follows along the lines of that of Theorem 1, with appropriate modifications that include substituting b for c, Theorem B for Theorem A, and replacing (6) and (7) with

$$(6') \qquad \sum_{n=1}^{m} 2nb^{2n-1} + \sum_{n=2m+3}^{\infty} nb^{n-1} \leq 1, \qquad m \geq 1.$$

$$(7') \qquad \sum_{n=2}^{m}(2n-1)b^{2n-2} + \sum_{n=2m+2}^{\infty} nb^{n-1} \leq \frac{1}{(1+b)^2}, \qquad m \geq 2.$$

References

1. J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. **17** (1915/16), 12–22.
2. P. L. Duren, *Univalent functions*, Springer-Verlag, New York, 1983.
3. R. Fournier, *On neighbourhoods of univalent starlike functions*, Ann. Polon. Math. **XLVII** (1986), 189–202.
4. A. W. Goodman and I. J. Schoenberg, *On a theorem of Szegö on univalent convex maps of the unit circle*, J. Analyse Math. **44** (1984/85), 200-204.
5. St. Ruscheweyh, *On the radius of univalence of the partial sums of convex functions*, Bull. London Math. Soc. **4** (1972), 367–369.
6. ——, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. **31** (1981), 521–527.
7. St. Ruscheweyh and T. Sheil-Small, *Hadamard products of schlicht functions and the Pólya-Schoenberg conjecture*, Comment. Math. Helv. **48** (1973), 119–135.
8. H. Silverman, *Radii problems for sections of convex functions*, Proc. Amer. Math. Soc. **104** (1988), 1191–1196.
9. G. Szegö, *Zur theorie der schlichten Abbildungen*, Math. Ann. **100** (1928), 188–211.

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