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STEENROD ALGEBRA MODULE MAPS FROM $H^*(B(\mathbf{Z}/p)^n)$ TO $H^*(B(\mathbf{Z}/p)^s)$

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ABSTRACT. Let $H^{\otimes n}$ denote the mod-p cohomology of the classifying space $B(\mathbf{Z}/p)^n$ as a module over the Steenrod algebra $\mathscr{A}$. Adams, Gunawardena, and Miller have shown that the $n \times s$ matrices with entries in $\mathbf{Z}/p$ give a basis for the space of maps $\operatorname{Hom}_{\mathscr{A}}(H^{\otimes n}, H^{\otimes s})$. For n and s relatively prime, we give a new basis for this space of maps using recent results of Campbell and Selick. The main advantage of this new basis is its compatibility with Campbell and Selick's direct sum decomposition of $H^{\otimes n}$ into $(p^n - 1)$ $\mathscr{A}$-modules.

Our applications are at the prime two. We describe the unique map from $\overline{H}$ to $D(n)$, the algebra of Dickson invariants in $H^{\otimes n}$, and we give the dimensions of the space of maps between the indecomposable summands of $H^{\otimes 3}$.

INTRODUCTION

Let $\mathscr{A}$ be the mod-p Steenrod algebra. And let H be the mod-p cohomology of the classifying space $B(\mathbf{Z}/p)$, so $H^{\otimes n} = H^*(B(\mathbf{Z}/p)^n; \mathbf{Z}/p)$. The set of degree-preserving $\mathscr{A}$-linear algebra maps from $H^{\otimes n}$ to $H^{\otimes s}$ is $M_{n,s}(\mathbf{Z}/p)$, the $n \times s$ matrices with entries in $\mathbf{Z}/p$. By a result of Adams, Gunawardena, and Miller, these algebra maps form a basis for the space of $\mathscr{A}$-module maps from $H^{\otimes n}$ to $H^{\otimes s}$ [AGM, LZ, Wo].

The main result of this paper is to give a new basis for the $\mathscr{A}$-module maps from $H^{\otimes n}$ to $H^{\otimes s}$ when n and s are relatively prime. Our maps are defined using Campbell and Selick's description of $H^{\otimes n}$ together with their decomposition of $H^{\otimes n}$ into a direct sum of $(p^n - 1)$ $\mathscr{A}$-modules [CS]. The main advantage of our new basis over the basis of algebra maps mentioned above is its compatibility with Campbell and Selick's decomposition, which although not complete is of considerably greater simplicity than the complete decomposition.

Most of the paper deals with the case $p = 2$. In §1, we recall Campbell and Selick's results and prove our main theorem (1.3). In §2, we recall some of the work of the first author and Kuhn regarding the complete decomposition of $H^{\otimes n}$. In §3, we show that there is a unique nonzero map from $\overline{H}$ to D_n, the algebra of Dickson invariants. We also determine the kernel and image of

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this map. In §4, we give the dimensions of the spaces of maps between the indecomposable summands of $H^{\otimes 3}$. These dimensions are also the entries in the Cartan matrix for the modular representations of $M_{3,3}(\mathbf{Z}/2)$. In §5, we show how our computations are related to the dimensions of certain Morava K-theories. In §6, we sketch the modifications necessary when p is an odd prime.

1

In this section, we recall Campbell and Selick's results and prove our main theorem.

Let $M_n = \mathbf{F}_2[x_i \mid i \in \mathbf{Z}/n]$ with $\deg x_i = 1$. Give M_n the structure of an $\mathscr{A}$-module by letting $\mathrm{Sq}^1(x_i) = x_{i-1}^2$ and using the Cartan formula on products. (One may check the Adem relations directly, or see [CS].) Note that M_n is an unstable $\mathscr{A}$-module and an $\mathscr{A}$-algebra but that M_n is not, in general, an unstable $\mathscr{A}$-algebra. The following theorem of Campbell and Selick provides the motivation and setting for this entire paper:

Theorem 1.1 [CS, 1]. *$M_n \cong H^{\otimes n}$ as $\mathscr{A}$-modules.*

Define weights $w(m) \in \mathbf{Z}/(2^n - 1)$ for monomials m in M_n by $w(1) = 0$, $w(x_i) = 2^i$, and $w(ab) = w(a) + w(b)$. For $j \in \mathbf{Z}/(2^n - 1)$, let $M_n(j)$ be the subspace spanned by the weight j monomials. Clearly, Sq^1 and hence all Steenrod operations preserve weight, so there is a decomposition $M_n \cong \bigoplus_{j \in \mathbf{Z}/(2^n-1)} M_n(j)$ as $\mathscr{A}$-modules.

Let $\pi_{n,j} : M_n \to M_n(j)$ and $\iota_{n,j} : M_n(j) \to M_n$ be the projection and inclusion. The algebra homomorphisms $\phi_{n,ns} : M_n \to M_{ns}$ and $\gamma_{ns,n} : M_{ns} \to M_n$ are defined on polynomial generators as specified by the following two equations:

$$\gamma_{ns,n}(x_i) = x_i, \qquad \phi_{n,ns}(x_i) = \sum_{j \equiv i \pmod n} x_j.$$

Definition 1.2. For $j \in \mathbf{Z}/(2^{ns} - 1)$, let $f_{n,s,j} : M_n \to M_s$ be the composite

$$M_n \xrightarrow{\phi_{n,ns}} M_{ns} \xrightarrow{\pi_{ns,j}} M_{ns}(j) \xrightarrow{\iota_{ns,j}} M_{ns} \xrightarrow{\gamma_{ns,s}} M_s.$$

Let $\overline{M}_n$ be the subspace of elements of degree greater than zero. Thus $\overline{M}_n(j) = M_n(j)$, for $j \neq 0$, and $\overline{M}_n(0)$ is the augmentation ideal in $M_n(0)$.

Theorem 1.3. *If $(n, s) = 1$, then the maps $f_{n,s,j}$, for $j \in \mathbf{Z}/(2^{ns} - 1)$, are linearly independent and give a basis for the space of maps from $\overline{M}_n$ to $\overline{M}_s$.*

Proof. First note that the previously known basis mentioned in the introduction tells us that if the maps are linearly independent, then they form a basis.

Fix k with $1 \leq k \leq (2^{ns} - 1)$. It suffices to show that $f_{n,s,j}(x_0^k)$ is the sum of x_0^k with other monomials if and only if $j = k$. Under the map $\phi_{n,ns}$, x_0^k

maps to

$$\sum_{i_0+i_1+\cdots+i_{s-1}=k} (i_0, i_1, \dots, i_{s-1}) x_0^{i_0} x_n^{i_1} \cdots x_{(s-1)n}^{i_{s-1}}, \tag{1.4}$$

where $(i_0, i_1, \dots, i_{s-1})$ is the multinomial coefficient. Since $(n, s) = 1$, $\gamma_{n,ns}$ sends the elements $x_0, x_n, \dots, x_{(s-1)n}$ in M_{ns} to some ordering of the generators $x_0, x_1, \dots, x_{s-1}$ in M_s. Hence, the only terms in (1.4) that project under $\gamma_{n,ns}$ to x_i^k for some i are $x_0^k, x_n^k, \dots$, and $x_{(s-1)n}^k$, and x_0^k is the only one of these that projects to x_0^k. The self-map $\iota_{ns,k} \circ \pi_{ns,k}$ of M_{ns} sends x_0^k to itself, so the image $f_{n,s,k}(x_0^k)$ contains x_0^k as a summand when written in the monomial basis. But, if $j \neq k$, then the self-map $\iota_{ns,j} \circ \pi_{ns,j}$ sends x_0^k to zero, so the image $f_{n,s,j}(x_0^k)$ does not contain x_0^k. □

Corollary 1.5. *If* $(n, s) = 1$, $i \in \mathbf{Z}/(2^n - 1)$, *and* $k \in \mathbf{Z}/(2^s - 1)$, *then*

$$\dim_{\mathbf{F}_2} \operatorname{Hom}_{\mathscr{A}}(\overline{M}_n(i), \overline{M}_s(k)) = \frac{(2^{ns} - 1)}{(2^n - 1)(2^s - 1)}.$$

Proof. The map $\pi_{s,k} \circ f_{n,s,j} \circ \iota_{n,i}$ is nonzero if and only if $j \equiv i \pmod{2^n - 1}$ and $j \equiv k \pmod{2^s - 1}$. The number of such j is $\frac{(2^{ns}-1)}{(2^n-1)(2^s-1)}$ by the Chinese Remainder Theorem, which applies by the following elementary observation. □

Lemma 1.6. *If* $(n, s) = 1$, *then* $(2^n - 1, 2^s - 1) = 1$.

Proof. Consider the set of relatively prime pairs (n, s) of positive integers ordered lexicographically. Suppose (n_0, s_0) is a minimal counterexample to the lemma (so $n_0 < s_0$). Then $(2^{n_0} - 1, 2^{s_0} - 1) = (2^{n_0} - 1, 2^{s_0} - 2^{n_0}) = (2^{n_0} - 1, 2^{s_0 - n_0} - 1)$, so $(n_0, s_0 - n_0)$ is a smaller counterexample. □

Remark 1.7. (1) Theorem 1.3 is false without the assumption $(n, s) = 1$. For example, $f_{2,2,3} = f_{2,2,6}$.

(2) For all n and s, any map $f : \overline{M}_n \to \overline{M}_s$ can be realized as a linear combination of compositions of maps of the form $f_{k,l,j}$. Since $H^{\otimes k}$ is a direct summand of $H^{\otimes(k+1)}$, $\overline{M}_k$ is isomorphic to a direct summand of $\overline{M}_{k+1}$. Hence, f factors as $\overline{M}_n \to \overline{M}_r \to \overline{M}_s$ for any r no smaller than either n or s. Our observation follows by choosing r to be relatively prime to both n and s.

We end this section with a related theorem due to the third author.

Theorem 1.8 [S1, 5.4, 5.6]. *For any* n *and* s, $\dim_{\mathbf{F}_2} \operatorname{Hom}_{\mathscr{A}}(\overline{M}_s(j), M_n) = \frac{2^{ns}-1}{2^s-1}$, *for* $j \in \mathbf{Z}/(2^s - 1)$.

Proof. First notice that $\operatorname{Hom}_{\mathscr{A}}(\overline{M}_s(j), M_n) \cong \operatorname{Hom}_{\mathscr{A}}(\overline{M}_s(j), H^{\otimes n})$. Let $\mathscr{U}$ be the category of unstable $\mathscr{A}$-modules and let $T : \mathscr{U} \to \mathscr{U}$ be the left adjoint to $\_\_ \otimes H$ defined by Lannes [L] (so $\operatorname{Hom}_{\mathscr{U}}(T(A), B) \cong \operatorname{Hom}_{\mathscr{U}}(A, B \otimes H)$). It is shown in [S1, 3.9] that $T(\overline{M}_s(j)) \cong \overline{M}_s(j) \oplus H^{\otimes s}$. It is then easy to calculate that

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{A}}(\overline{M}_s(j), H^{\otimes n}) &\cong \mathrm{Hom}_{\mathscr{A}}(T\overline{M}_s(j), H^{\otimes(n-1)}) \\
&\cong \mathrm{Hom}_{\mathscr{A}}(\overline{M}_s(j), H^{\otimes(n-1)}) \oplus \mathrm{Hom}_{\mathscr{A}}(H^{\otimes s}, H^{\otimes(n-1)}) \\
&\ \ \vdots \\
&\cong \mathrm{Hom}_{\mathscr{A}}(\overline{M}_s(j), \mathbf{F}_2) \\
&\quad \oplus \mathrm{Hom}_{\mathscr{A}}(H^{\otimes s}, \mathbf{F}_2) \oplus \cdots \oplus \mathrm{Hom}_{\mathscr{A}}(H^{\otimes s}, H^{\otimes(n-1)}).
\end{aligned}$$

The dimension of the last line is $0 + 1 + 2^s + \cdots + 2^{s(n-1)} = \frac{2^{ns}-1}{2^s-1}$. □

2

In this section, we recall some of the work of the first author and Kuhn regarding the complete decomposition of $H^{\otimes n}$. We begin with the relevant representation theory (see [HK, §6]).

There are 2^n distinct irreducible $\mathbf{F}_2$-representations of the ring $R = \mathbf{F}_2[M_{n,n}(\mathbf{Z}/2)]$, which are denoted by $\{S_{\lambda_1, \lambda_2, \dots, \lambda_n} | 0 \le \lambda_i \le 1\}$. ($S_{(\lambda)}$ is the top quotient of the Weyl module W^α associated to the partition $(\alpha_1, \dots, \alpha_n)$, where $\lambda_i = \alpha_i - \alpha_{i+1}$.) Let $P_{(\lambda)}$ be the projective cover of $S_{(\lambda)}$ (i.e., the smallest projective representation with a surjection onto $S_{(\lambda)}$), and let $e_{(\lambda)}$ be a primitive idempotent in R with $P_{(\lambda)} \cong Re_{(\lambda)}$.

For $(\mu) = (\mu_1, \mu_2, \dots, \mu_n)$, with $0 \le \mu_i \le 1$, let $c_{\mu\lambda}$ be the number of times $S_{(\mu)}$ occurs in a composition series for $P_{(\lambda)}$. The $2^n \times 2^n$ matrix $(c_{\mu\lambda})$ is called the Cartan matrix for R. In the following proposition, we summarize the relevant results:

Proposition 2.1.

(i) [HK, 4.1] $R \cong \bigoplus_{(\lambda)} \dim_{\mathbf{F}_2}(S_{(\lambda)}) P_{(\lambda)}$.
(ii) [CR, 54.15, 54.16] $c_{\mu\lambda} = \dim_{\mathbf{F}_2} \mathrm{Hom}_R(P_{(\mu)}, P_{(\lambda)})$.
(iii) [K3] $c_{\mu\lambda} = c_{\lambda\mu}$.
(iv) [K4] *Let* $\lambda = (1, \dots, 1)$. *Then* $P_{(\lambda)}$ *is both projective and irreducible, so* $c_{\mu\lambda}$ *is* 1, *if* $\mu = \lambda$, *and* 0 *otherwise.*

Remark 2.2. (1) Parts (i) and (ii) follow from the fact that the $S_{(\lambda)}$ are absolutely irreducible and $\mathbf{F}_2$ is a splitting field for R.

(2) Part (i) is one of the main ingredients in the proof of Theorem 2.3 below.

(3) The symmetry of the Cartan matrix is standard for group rings, but not generally true for semigroup rings.

(4) We thank Kuhn for showing us the following proof of part (iv): Let e be the Steinberg idempotent $\overline{\Sigma}_{n+1}\overline{B}_{n+1}/[\mathrm{GL}_{n+1} : U_{n+1}]$ in $\mathbf{F}_2[\mathrm{GL}_{n+1}(\mathbf{Z}/2)]$ [MP, 2.6]. Then $A \cdot e = 0$ for any matrix A with rank $\le n-1$ [K1, 2.3(1)]. There is a primitive orthogonal idempotent decomposition $e = e_{n+1} + e_n$ in

$\mathbf{F}_2[M_{n+1,n+1}(\mathbf{Z}/2)]$, and the idempotent e_n is conjugate to an idempotent $e_{(\lambda)}$ in $\mathbf{F}_2[M_{n,n}(\mathbf{Z}/2)]$ inducing $P_{(\lambda)}$. It follows that $A \cdot e_{(\lambda)} = 0$ for any matrix A with rank $\leq n-1$, so $\mathbf{F}_2[M_{n,n}(\mathbf{Z}/2)]e_{(\lambda)} = \mathbf{F}_2[\mathrm{GL}_n(\mathbf{Z}/2)]e_{(\lambda)}$. This vector space is irreducible as a $\mathrm{GL}_n(\mathbf{Z}/2)$-module (since it is isomorphic to the Steinberg representation), so it is irreducible as an $M_{n,n}(\mathbf{Z}/2)$-module.

Now we relate the above results to the $\mathscr{A}$-module decompositions of $H^{\otimes n}$. The ring R acts on the stable classifying space $B(\mathbf{Z}/2)^n_+$, and $X_{(\lambda)} = e_{(\lambda)}B(\mathbf{Z}/2)^n_+$ denotes the stable wedge summand associated to (λ).

Theorem 2.3 [HK, A, §6]. *The following two decompositions are complete*:

(i) $B(\mathbf{Z}/2)^n_+ \simeq \bigvee_{0\leq\lambda_i\leq 1} \dim_{\mathbf{F}_2}(S_{\lambda_1,\dots,\lambda_n})X_{\lambda_1,\dots,\lambda_n}$, *as spectra.*

(ii) $H^{\otimes n} \cong \bigoplus_{0\leq\lambda_i\leq 1} \dim_{\mathbf{F}_2}(S_{\lambda_1,\dots,\lambda_n})H^*(X_{\lambda_1,\dots,\lambda_n})$, *as $\mathscr{A}$-modules.*

Remark 2.4. $X_{\lambda_1,\dots,\lambda_{n-1},0} \simeq X_{\lambda_1,\dots,\lambda_{n-1}}$, so each wedge summand (resp. $\mathscr{A}$-module summand) of $B(\mathbf{Z}/2)^{n-1}_+$ (resp. $H^{\otimes(n-1)}$) appears as a well-identified summand of $B(\mathbf{Z}/2)^n_+$ (resp. $H^{\otimes n}$).

We now state a special case of the theorem of Adams, Gunawardena, and Miller referred to in the introduction.

Theorem 2.5 [AGM, p. 438]. $R = \mathbf{F}_2[M_{n,n}(\mathbf{Z}/2)] \cong \mathrm{Hom}_{\mathscr{A}}(H^{\otimes n}, H^{\otimes n})$.

Corollary 2.6. $e_{(\mu)}Re_{(\lambda)} \cong \mathrm{Hom}_R(P_{(\mu)}, P_{(\lambda)}) \cong \mathrm{Hom}_{\mathscr{A}}(\widetilde{H}^*(X_{(\mu)}), \widetilde{H}^*(X_{(\lambda)}))$, *as vector spaces.*

Remark 2.7. (1) By Proposition 2.1(ii), this corollary relates the Cartan invariants $c_{\mu\lambda}$ to the dimensions of the maps between the indecomposable $\mathscr{A}$-module summands.

(2) Note that

$$\dim_{\mathbf{F}_2}(P_{(\lambda)}) = \dim_{\mathbf{F}_2} \mathrm{Hom}_R(R, P_{(\lambda)}) = \dim_{\mathbf{F}_2} \mathrm{Hom}_{\mathscr{A}}(H^{\otimes n}, \widetilde{H}^*(X_{(\lambda)})).$$

By definition, the first of these is equal to $\sum_{(\mu)} \dim_{\mathbf{F}_2}(S_{(\mu)})c_{\mu\lambda}$. These "weighted column sums" will be used in §4.

Now let $\overline{Y}_n(j)$ denote the stable wedge summand of $B(\mathbf{Z}/2)^n_+$ corresponding to $\overline{M}_n(j)$ (i.e., $\widetilde{H}^*(\overline{Y}_n(j)) \cong \overline{M}_n(j)$). The complete decompositions of the $\overline{Y}_n(j)$ are described by the first author in [H]. The next proposition will be used in §3.

Proposition 2.8 [H, 4.12]. *Let* $X(m) = X_{0,\dots,0,1,0,\dots,0}$, *where the* 1 *is in the* m*th position. Then* $X(\alpha(j))$ *is a summand of* $\overline{Y}_n(j)$, *for* $1 \leq j \leq (2^n-1)$, *where* $\alpha(j)$ *is the number of* 1*'s in the dyadic expansion of* j.

Note $X(m)$ is a summand of $B(\mathbf{Z}/2)^n_+$ for each $n \geq m$, by Remark 2.4.

3

Here we study the properties of the unique nonzero map $f_{1,n,k}$ from $\overline{M}_1$ to $\overline{M}_n(k)$ and give an application regarding the Dickson invariants. First we determine the indecomposable summand of $\overline{M}_n(k)$ in which the image of this map lies.

Theorem 3.1. *The image of* $\overline{M}_1 \to \overline{M}_n(k)$, $1 \leq k \leq (2^n - 1)$, *lies in* $\widetilde{H}^*(X(\alpha(k)))$.

Proof. By induction on n and Proposition 2.8, the statement is true for $1 \leq k < (2^n - 1)$. (Of course, when $n = 1$, $\overline{M}_1$ is indecomposable and isomorphic to $\widetilde{H}^*(X(1))$.) From Theorem 2.3 and the known dimensions of the irreducibles $S_{0,\dots,0,1,0,\dots,0}$ [JK, 8.3.9], it follows that

$$B(\mathbf{Z}/2)^n \simeq \binom{n}{1} X(1) \vee \binom{n}{2} X(2) \vee \cdots \vee \binom{n}{n-1} X(n-1) \vee X(n) \vee Z,$$

with Z a wedge of other indecomposables each with multiplicity greater than one. Since $\overline{M}_1$ has $(2^n - 1)$ linearly independent maps to $\overline{M}_n$, it follows that there must be a map $\overline{M}_1 \to \widetilde{H}^*(X(n))$. □

Theorem 3.2. *The kernel of the map* $\overline{M}_1 \to \overline{M}_n(k)$, $1 \leq k \leq (2^n - 1)$, *has basis* $\{x_0^j | \alpha(j) < \alpha(k)\}$.

By Theorem 3.1, the map $\overline{M}_1 \to \overline{M}_n(k)$ has the same kernel as the map $\overline{M}_1 \to \overline{M}_{\alpha(k)}(0)$, so it suffices to prove the following special case:

Theorem 3.3. *The kernel of the map* $\overline{M}_1 \to \overline{M}_n(0)$ *has basis* $\{x_0^j | \alpha(j) < n\}$.

Proof. Write $j = (j_m j_{m-1} \cdots j_0)$ in its dyadic expansion ($j_i = 0$ or 1). Then

$$\begin{aligned} x_0^j &\mapsto (x_0 + \cdots + x_{n-1})^j \\ &= (x_0^{2^m} + \cdots + x_{n-1}^{2^m})^{j_m} (x_0^{2^{m-1}} + \cdots + x_{n-1}^{2^{m-1}})^{j_{m-1}} \cdots (x_0 + \cdots + x_{n-1})^{j_0} \end{aligned} \tag{3.4}$$

under $\overline{M}_1 \to \overline{M}_n$.

In each of the polynomials $(x_0^{2^l} + \cdots + x_{n-1}^{2^l})$, the terms have weights $1, 2, \dots, 2^{n-1}$ modulo $(2^n - 1)$ (after some reordering). When the product (3.4) is expanded, there are $n^{\alpha(j)}$ distinct terms, each with a weight that is a sum of $\alpha(j)$ 2-powers. Also, each set of $\alpha(j)$ 2-powers occurs at least once.

(i) If $\alpha(j) < n$, then none of the monomials in (3.4) has weight congruent to zero modulo $(2^n - 1)$.

(ii) If $\alpha(j) \geq n$, then at least one monomial in (3.4) has weight zero. This follows from the fact that any number modulo $(2^n - 1)$ can be written in at least one way as a sum of α 2-powers if $\alpha \geq n$ (e.g., $1 \equiv 2^{n-1} + 2^{n-1} \equiv 2^{n-1} + 2^{n-2} + 2^{n-2} \equiv$, etc. (mod $2^n - 1$)). □

It is well known and not hard to see that the elements $\{x_0^{2^l - 1} | l = 1, 2, \dots\}$

generate $\overline{M}_1$ as an $\mathscr{A}$-module. We can restate Theorem (3.3) as follows:

Corollary 3.5. *The map $\overline{M}_1 \to \overline{M}_n(0)$ has coimage*

$$\overline{M}_1/\mathscr{A}\langle x_0^1, x_0^3, \dots, x_0^{2^{n-1}-1}\rangle .$$

Now we identify the map $\overline{M}_1 \to \overline{M}_n(0)$ in the basis of algebra maps [AGM]. Recall that the $\mathscr{A}$-algebra maps from H to $H^{\otimes n}$ are given by the matrices $M_{1,n}(\mathbf{Z}/2)$. Write $H^{\otimes n} = \mathbf{F}_2[t_0, \dots, t_{n-1}]$ with the usual Steenrod algebra action $\mathrm{Sq}^1(t_i) = t_i^2$. Recall that the Dickson algebra, D_n, the subalgebra of $H^{\otimes n}$ which is fixed under the $\mathrm{GL}_n(\mathbf{Z}/2)$ action, is a polynomial algebra on elements a_i with $\deg(a_i) = 2^n - 2^{n-i}$ [D, Wi].

Definition 3.6. Let $\Phi : H \to H^{\otimes n}$ be the sum of all of the $\mathscr{A}$-algebra maps. That is, $\Phi(1) = 2^n = 0$, and, for $i > 0$,

$$\Phi(t_0^i) = 0^i + t_0^i + t_1^i + \cdots + t_{n-1}^i + (t_0 + t_1)^i + \cdots + (t_0 + \cdots + t_{n-1})^i .$$

Theorem 3.7. *The map from $\overline{M}_1$ to $\overline{M}_n$ corresponding to $\Phi : H \to H^{\otimes n}$ is $f_{1,n,0}$.*

Proof. The symmetric group Σ_{2^n-1} acts on the set of nonzero degree-one elements of $H^{\otimes n}$. The image of Φ is invariant under this action. Since $\mathrm{GL}_n(\mathbf{Z}/2) \subseteq \Sigma_{2^n-1}$, this image is contained in the Dickson invariants, D_n.

Selick and Campbell show that there is an isomorphism of $\mathscr{A}$-modules, $M_n(0) \cong (H^{\otimes n})^T$, where T is a certain matrix in $\mathrm{GL}_n(\mathbf{Z}/2)$. So the image of the map from $\overline{M}_1 \to \overline{M}_n$ corresponding to Φ must be contained in $M_n(0)$ (actually in $\overline{M}_n(0)$ since $\Phi(1) = 0$). The only such map is $f_{1,n,0}$. □

By Theorem (3.3), the least-degree element in $\overline{M}_1$ with a nonzero image under $f_{1,n,0}$ is $x_0^{2^n-1}$. Therefore Φ is nonzero on $t_0^{2^n-1}$. Indeed, $t_0^{2^n-1}$ hits the top polynomial generator, a_n, for the Dickson algebra (since a_n is the only element with degree $2^n - 1$).

Corollary 3.8. *A $\mathscr{A}$-submodule of $H/\mathscr{A}\langle t_0^1, t_0^3, \dots, t_0^{2^{n-1}-1}\rangle$ generated by $t_0^{2^n-1}$ is isomorphic to the $\mathscr{A}$-submodule of the Dickson invariants D_n generated by a_n.*

Remark 3.9. As mentioned above, the isomorphism $H^{\otimes n} \cong M_n$ is as $\mathscr{A}$-modules, but not as rings. It has been shown by the second author and Selick that, when restricted to the Dickson invariants, this map is multiplicative. Hence $M_n(0)$ contains a polynomial subalgebra, as does $(H^{\otimes s})^T$.

4

In this section, we give the dimensions of the spaces of $\mathscr{A}$-module maps between the indecomposable summands of $H^{\otimes 3}$. By Corollary 2.6, these dimensions are the entries in the Cartan matrix for $\mathbf{F}_2[M_{3,3}(\mathbf{Z}/2)]$. We should mention that the Cartan matrix for $\mathbf{F}_p[M_{2,2}(\mathbf{Z}/p)]$ is known for all p by work of Glover [G].

Most of the wedge summands $X_{\lambda_1,\lambda_2,\lambda_3}$ of $B(\mathbf{Z}/2)^3_+$ have been identified in more familiar terms as follows: $X_{0,0,0} \simeq S^0$, $X_{0,0,1} \simeq BG_{168}$, $X_{0,1,0} \simeq BA_4$, $X_{1,0,0} \simeq B\mathbf{Z}/2$, $X_{1,1,0} \simeq L(2)$, and $X_{1,1,1} \simeq L(3)$, where G_{168} is a semidirect product of $(\mathbf{Z}/2)^3$ with an order-21 subgroup of $\mathrm{GL}_3(\mathbf{Z}/2)$, A_4 is the alternating group on 4 letters, and $L(n) = \Sigma^{-n}(Sp^{2^n}(S^0)/Sp^{2^{n-1}}(S^0))$. (Note that $L(1) \simeq B\mathbf{Z}/2$.) We use Y for $X_{0,1,1}$ and W for $X_{1,0,1}$. (We assume all spectra have been localized at 2.)

By abuse of notation, for the rest of this section we will use the notations S^0, BG_{168}, etc., for the cohomologies of these spectra: $\tilde{H}^*(S^0)$, $\tilde{H}^*(BG_{168})$, etc. The weight summands in $H^{\otimes n}$ for $n \leq 3$ decompose as follows:

Proposition 4.1 [H, 6.1].

$$\begin{aligned}
M_1(0) &\cong S^0 \oplus B\mathbf{Z}/2,\\
M_2(0) &\cong S^0 \oplus BA_4,\\
M_2(1) \cong M_2(2) &\cong B\mathbf{Z}/2 \oplus L(2),\\
M_3(0) &\cong S^0 \oplus BG_{168} \oplus 2L(2) \oplus 2L(3),\\
M_3(1) \cong M_3(2) \cong M_3(4) &\cong B\mathbf{Z}/2 \oplus W \oplus L(2) \oplus L(3),\\
M_3(3) \cong M_3(5) \cong M_3(6) &\cong BA_4 \oplus Y \oplus L(2) \oplus L(3).
\end{aligned}$$

Theorem 4.2. *The dimensions* $\dim_{\mathbf{F}_2} \mathrm{Hom}_{\mathscr{A}}(\tilde{H}^*(X_{(\mu)}), \tilde{H}^*(X_{(\lambda)}))$ *for* $X_{(\mu)}$, $X_{(\lambda)}$ *summands of* $B(\mathbf{Z}/2)^3_+$ *are given by the following array*:

	S^0	BG_{168}	$B\mathbf{Z}/2$	W	BA_4	Y	$L(2)$	$L(3)$
S^0	1	0	0	0	0	0	0	0
BG_{168}	0	5	1	3	3	1	0	0
$B\mathbf{Z}/2$	0	1	1	0	1	0	0	0
W	0	3	0	6	2	3	1	0
BA_4	0	3	1	2	3	0	0	0
Y	0	1	0	3	0	4	1	0
$L(2)$	0	0	0	1	0	1	1	0
$L(3)$	0	0	0	0	0	0	0	1

Proof. This matrix is symmetric by Proposition 2.1(iii).

(i) S^0 is concentrated in dimension zero, so the S^0 row is easy.

(ii) The $B\mathbf{Z}/2$ row follows from Corollary 1.5.

(iii) The $L(3)$ row follows from Proposition 2.1(iv).

(iv) Using maps from $\overline{M}_2$ to $\overline{M}_2$, we have the submatrix

	BA_4	$B\mathbf{Z}/2$	$L(2)$
1 BA_4	a	1	0
2 $B\mathbf{Z}/2$	1	1	0
2 $L(2)$	0	0	1
	5	3	2

where the left-hand column of numbers denotes multiplicities in $H^{\otimes 2}$ and the bottom row of numbers denotes $\dim_{\mathbf{F}_2} \operatorname{Hom}_{\mathscr{A}}(M_2, \tilde{H}^*(X_{(\lambda)}))$. Since $BA_4 \cong \overline{M}_2(0)$, its weighted column sum (see Remark 2.7) is $\frac{2^4-1}{2^2-1} = 5$ by Theorem 1.8. Hence the unknown a must be 3. (Alternatively, one can show directly that the three self-maps $f_{3,2,3} \circ f_{2,3,3}$, $f_{3,2,24} \circ f_{2,3,3}$, and $f_{3,2,45} \circ f_{2,3,3}$ of $\overline{M}_2(3)$ are linearly independent.)

(v) Now consider the submatrix giving maps $\overline{M}_2$ to $\overline{M}_3$:

	BG_{168}	$B\mathbf{Z}/2$	W	BA_4	Y	$L(2)$	$L(3)$
BA_4	b	1	c	3	d	0	0
$B\mathbf{Z}/2$	1	1	0	1	0	0	0
$L(2)$	e	0	f	0	g	1	0

Using Theorem 1.3, we have $\dim_{\mathbf{F}_2} \operatorname{Hom}_{\mathscr{A}}(\overline{M}_2(0), \overline{M}_3(0)) = 3$. Using Proposition 4.1, we then have $b + 2 \cdot 0 + 2 \cdot 0 = 3$, so $b = 3$. Similarly, $\dim_{\mathbf{F}_2} \operatorname{Hom}_{\mathscr{A}}(\overline{M}_2(0), \overline{M}_3(1)) = 3$, so $1+c+0+0=3$, and $c=2$. The values $d=0$, $e=0$, $f=1$, and $g=1$ are found the same way.

(vi) The remaining unknowns are indicated in the following table:

	BG_{168}	$B\mathbf{Z}/2$	W	BA_4	Y	$L(2)$	$L(3)$
1 BG_{168}	h	1	i	3	j	0	0
3 $B\mathbf{Z}/2$	1	1	0	1	0	0	0
3 W	i	0	k	2	l	1	0
3 BA_4	3	1	2	3	0	0	0
3 Y	j	0	l	0	m	1	0
8 $L(2)$	0	0	1	0	1	1	0
8 $L(3)$	0	0	0	0	0	0	1
	n	7	p	21	q	14	8

Again, in this table, the left-hand column denotes multiplicities in $\overline{M}_3$ and the bottom row gives $\dim_{\mathbf{F}_2} \operatorname{Hom}_{\mathscr{A}}(\overline{M}_3, \tilde{H}^*(X_{(\lambda)}))$.

By Theorem 1.8, $\dim_{\mathbf{F}_2} \operatorname{Hom}_{\mathscr{A}}(\overline{M}_3, \overline{M}_3(0)) = 73$, so $n + 2(14) + 2(8) = 73$ (by Proposition 4.1), and $n = 29$. Similarly, $p = 44$ and $q = 30$.

From the weighted column sums (see Remark 2.7),

(1) $h + 3 + 3i + 9 + 3j = 29$, so $h + 3i + 3j = 17$;
(2) $i + 3k + 6 + 3l + 8 = 44$, so $i + 3k + 3l = 30$; and
(3) $j + 3l + 3m + 8 = 30$, so $j + 3l + 3m = 22$.

It follows from (1) and (2) that $i = 0$ or 3. We show below that $i > 0$, so $i = 3$. This, (1), and (3) imply $j = 1$. Then (1) implies $h = 5$.

To show $i > 0$, we need to produce a nonzero map from BG_{168} to W. Consider the compositions $\overline{M}_3(0) \to \overline{M}_2(0) \to \overline{M}_3(1)$. Any such composition must be zero on $2L(2) \oplus 2L(3)$ and cannot hit $L(2) \oplus L(3)$, since the middle term is BA_4. Hence any such composition gives a map $BG_{168} \to B\mathbf{Z}/2 \oplus W$. The two compositions $f_{2,3,15} \circ f_{3,2,21}$ and $f_{2,3,36} \circ f_{3,2,21}$ are linearly independent (their images on $x_0^4 x_1^3 x_2$ differ). Since there is only one nonzero map from BG_{168} to $B\mathbf{Z}/2$, either one of the above compositions or their sum is a map from BG_{168} to W.

(vii) To determine k, l, and m, we construct linearly independent maps $\overline{M}_3(1) \to \overline{M}_3(1)$ and $\overline{M}_3(1) \to \overline{M}_3(3)$ via compositions through $\overline{M}_4(1)$. From what we already know,

$$\dim_{\mathbf{F}_2} \operatorname{Hom}_{\mathscr{A}}(\overline{M}_3(1), \overline{M}_3(1)) = 5 + k,$$
$$\dim_{\mathbf{F}_2} \operatorname{Hom}_{\mathscr{A}}(\overline{M}_3(1), \overline{M}_3(3)) = 7 + l,$$
$$\dim_{\mathbf{F}_2} \operatorname{Hom}_{\mathscr{A}}(\overline{M}_3(3), \overline{M}_3(3)) = 7 + m, \qquad k + l = 9, \quad \text{and} \quad l + m = 7.$$

The following lemma shows that $k \geq 6$ and $l \geq 3$, so $k = 6$, $l = 3$, and $m = 4$. □

Lemma 4.3. *There are at least eleven linear independent maps $\overline{M}_3(1) \to \overline{M}_3(1)$ and at least ten linearly independent maps $\overline{M}_3(1) \to \overline{M}_3(3)$.*

Proof. For simplicity, let $g_j = f_{4,3,j} \circ f_{3,4,1}$.

(i) The maps g_1, g_{106}, g_{526}, g_{736}, g_{1156}, g_{1576}, and g_{2626} have linearly independent images on x_0^{15}. The maps g_{316}, g_{631}, g_{1891}, and g_{2206} are zero on x_0^{15} and have linearly independent images on $x_0^7 x_1^3 x_2^{11}$.

(ii) The maps g_{136}, g_{556}, g_{1186}, g_{2656}, and g_{3076} have linearly independent images on x_0^{15}. The maps g_{346}, g_{871}, g_{1606}, g_{1816}, and g_{2236} are zero on x_0^{15} and have linearly independent images on $x_0^7 x_1^3 x_2^{11}$. □

Remark 4.4. (1) We note that part (iv) of the proof of Theorem 4.2 and part (i) of the proof of Lemma 4.3 show that $f_{2,3,3}$ and $f_{3,4,1}$ are injections, and therefore split. We would not be surprised to find that $f_{n,n+1,j} : \overline{M}_n(j) \to \overline{M}_{n+1}(j)$, $1 \leq j \leq (2^n - 1)$, is always an injection. (Note that $\pi_{ns,j} \circ \phi_{n,ns} \circ \iota_{n,j} : \overline{M}_n(j) \to \overline{M}_{ns}(j)$ is an injection [CS, 4].)

(2) Paul Selick has observed that the weighted sums of the diagonal elements $(\sum_{(\lambda)} \dim(S_{(\lambda)}) c_{\lambda\lambda})$ for $n = 1, 2$, and 3 are 2, 8, and 64. We do not know whether this pattern $(n \mapsto 2^{\binom{n+1}{2}})$ continues.

5

In this section, we show how our computations are related to the dimensions of certain Morava K-theories. The following theorem provides the connection between these dimensions and the $\mathscr{A}$-module maps. For X any (2-local) spectrum, let $k_n(X) = \dim_{K(n)_*} K(n)_*(X)$, where $K(n)_*$ is the nth Morava K-theory.

Theorem 5.1 [K2, 1.7]. *If X is a stable wedge summand of $B(\mathbf{Z}/2)^s_+$ for any s, then $k_n(X) = \dim_{\mathbf{F}_2} \operatorname{Hom}_{\mathscr{A}}(H^{\otimes n}, \widetilde{H}^*(X))$.*

For example, the numbers at the bottom of the table in part (vi) of the proof of Theorem 4.2 give values of k_3.

Combining Theorems 5.1 and 1.8, we obtain the following.

Theorem 5.2 [S1, 5.7]. $k_n(\overline{Y}_s(j)) = \frac{2^{ns}-1}{2^s-1}$.

The k_n's for the wedge summands of $B(\mathbf{Z}/2)^3_+$ can now be calculated using Theorems 5.2 and 4.1 and the next proposition.

Proposition 5.3 [K2, 5.2(1), 6.5]. $k_n(L(m)) = \frac{(2^n-1)(2^n-2)\cdots(2^n-2^{m-1})}{1\cdot 3\cdots(2^m-1)}$.

Theorem 5.4.

$$\begin{aligned} k_n(BA_4) &= \frac{1}{3}(2^n-1)(2^n+1), \\ k_n(BG_{168}) &= \frac{1}{21}(2^n-1)(2^{2n}+2^n+15), \\ k_n(W) &= \frac{2}{21}(2^n-1)(2^n-2)(2^n+3), \\ k_n(Y) &= \frac{2}{21}(2^n-1)(2^n-2)(2^n-\frac{1}{2}). \end{aligned}$$

Remark 5.5. The methods in [HKR] can be used to find $k_n(BG)$ for any finite group G, so could be used for BA_4 and BG_{168}.

6

In this section, we sketch the modifications necessary when p is an odd prime. Let $H^{\otimes n} = H^*(B(\mathbf{Z}/p)^n; \mathbf{F}_p)$ as in the introduction. Let M_n be the free graded commutative algebra on $\{y_i | i \in \mathbf{Z}/n\}$ and $\{x_i | i \in \mathbf{Z}/n\}$ where the y's have degree 1 and the x's have degree 2. Give M_n the Steenrod algebra action extending $\beta(y_i) = x_i$ and $\mathscr{P}^1(x_i) = x_{i-1}^p$.

Theorem 6.1 [CS, §3]. *$H^{\otimes n} \cong M_n$ as $\mathscr{A}$-modules.*

Define weights $w(m) \in \mathbf{Z}/(p^n-1)$ by $w(y_i) = w(x_i) = p^i$ and $w(ab) = w(a) + w(b)$. Define the weight summands $M_n(j)$ and the projections and inclusions $\pi_{n,j}$, $\iota_{n,j}$ as before. Define the maps $\phi_{n,ns}$ and $\gamma_{ns,n}$ as before and extend to the y's by the same formulas. Finally, define $f_{n,s,j} : M_n \to M_s$ for $j \in \mathbf{Z}/(p^{ns}-1)$ as in Definition 1.2.

The proofs of the following are essentially the same as in §1:

Theorem 6.2. *If* $(n, s) = 1$, *then the maps* $f_{n,s,j}$, *for* $j \in \mathbf{Z}/(p^{ns} - 1)$, *are linearly independent and give a basis for the space of maps from* $\overline{M}_n$ *to* $\overline{M}_s$.

Corollary 6.3. *If* $(n, s) = 1$, $i \in \mathbf{Z}/(p^n - 1)$, *and* $k \in \mathbf{Z}/(p^s - 1)$, *then*

$$\dim_{\mathbf{F}_p} \operatorname{Hom}_{\mathscr{A}}(\overline{M}_n(i), \overline{M}_s(k)) = \begin{cases} (p-1)\dfrac{(p^{ns}-1)}{(p^n-1)(p^s-1)}, & \text{if } i \equiv k \pmod{p-1}; \\ 0, & \text{otherwise}. \end{cases}$$

Lemma 6.4. *If* $(n, s) = 1$, *then* $(p^n - 1, p^s - 1) = (p - 1)$.

Theorem 6.5. *For any* n *and* s, $\dim_{\mathbf{F}_p} \operatorname{Hom}_{\mathscr{A}}(\overline{M}_s(j), M_n) = \frac{p^{ns}-1}{p^s-1}$, *for* $j \in \mathbf{Z}/(p^s - 1)$.

We suspect that the results in §3 also have odd-primary analogs. For example, the analog of Theorem 3.3 says that the kernel of the map $\overline{M}_1(0) \to M_n(0)$ has basis $\{x_0^i, x_0^{i-1}y_0 | i > 0, \ i \equiv 0 \pmod{p-1}, \ \alpha(i) < n(p-1)\}$. (Note that $M_1(0)$ has basis $\{x_0^i, x_0^{i-1}y_0 | i \equiv 0 \pmod{p-1}\}$.)

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References

[AGM] J. F. Adams, J. H. Gunawardena, and H. R. Miller, *The Segal conjecture for elementary abelian p-groups*, Topology **24** (1985), 435–460.

[CR] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Wiley, New York, 1962.

[CS] H. E. A. Campbell and P. S. Selick, *Polynomial algebras over the Steenrod algebra*, Comment. Math. Helv. **65** (1990), 171–180.

[D] L. E. Dickson, *A fundamental system of invariants of the general modular linear group with a solution to the form problem*, Trans. Amer. Math. Soc. **12** (1911), 75–98.

[G] D. J. Glover, *A study of certain modular representations*, J. Algebra **51** (1978), 425–475.

[H] J. C. Harris, *On certain stable wedge summands of* $B(\mathbf{Z}/p)_+^n$, Canad. J. Math. (to appear).

[HK] J. C. Harris and N. J. Kuhn, *Stable decompositions of classifying spaces of finite abelian p-groups*, Math. Proc. Cambridge Philos. Soc. **103** (1988), 427–449.

[HKR] M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel, *Generalized group characters and complex oriented cohomology theories*, preprint, 1989.

[JK] G. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia Math. Appl., vol. 16, Addison-Wesley, Reading, Mass., 1981.

[K1] N. J. Kuhn, *The rigidity of* $L(n)$, in Algebraic Topology—Proc., Seattle, 1985, Lecture Notes in Math., vol. 1286, Springer-Verlag, Berlin, 1987, pp. 286–292.

[K2] ——, *The Morava K-theory of some classifying spaces*, Trans. Amer. Math. Soc. **304** (1987), 193–205.

[K3] ——, *Transpose duality and Carlsson's injectives*, in preparation.

[K4] ——, private correspondence.

[L] J. Lannes, *Sur la cohomologie modulo p des p-groups abelians elementaires*, Proc. Durham Symposium on Homotopy Theory 1985, L. M. S. Lecture Notes, vol. 117, Cambridge Univ. Press, Cambridge, 1987, pp. 97–116.

[LZ] J. Lannes and S. Zarati, *Sur les foncteurs dérivés de la déstabilisation*, Math. Z. **194** (1987), 25–59.

[MP] S. A. Mitchell and S. B. Priddy, *Stable splittings derived from the Steinberg module*, Topology **22** (1983), 285–298.

[S1] R. J. Shank, *Polynomial algebras over the Steenrod algebra, summands of* $H^*(B(\mathbf{Z}/2\mathbf{Z})^s)$ *and Lannes' division functors*, Ph.D. thesis, University of Toronto, 1989.

[S2] ——, *Symmetric algebras over the Steenrod algebra and Lannes' T functor*, preprint, 1989.

[Wi] C. Wilkerson, *A primer on the Dickson invariants*, Proc. Northwestern Homotopy Theory Conference 1982, Contemp. Math., vol. 19, Amer. Math. Soc., Providence, R.I., 1983, pp. 421–434.

[Wo] R. M. W. Wood, *Splitting* $\Sigma(\mathbf{C}P^\infty \times \cdots \times \mathbf{C}P^\infty)$ *and the action of Steenrod squares* Sq^i *on the polynomial ring* $\mathbf{F}_2[x_1, \ldots, x_n]$, in Algebraic Topology—Barcelona, 1986, Lecture Notes in Math., vol. 1298, Springer, Berlin, 1987, pp. 237–255.

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