A CERTAIN CLASS OF TRIANGULAR ALGEBRAS
IN TYPE II₁ HYPERFINITE FACTORS

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Abstract. Let \( \mathcal{T} \) be the standard triangular UHF algebra in a UHF algebra \( \mathcal{A} \), where the rank of \( \mathcal{A} \) is a strictly increasing sequence of positive integers. Let \( \mathcal{M} \) be the type II₁ hyperfinite factor defined as the weak closure of \( \mathcal{T} \) in the tracial representation of \( \mathcal{A} \). Define \( \mathcal{I} \) to be the weak closure of \( \mathcal{T} \) in this representation. Then \( \mathcal{I} \) is a reflexive, maximal weakly closed triangular algebra in \( \mathcal{M} \). Moreover, \( \mathcal{I} \) is irreducible relative to \( \mathcal{M} \). We exhibit a strongly closed sublattice \( \mathcal{L} \) of \( \text{lat}\mathcal{I} \) such that \( \mathcal{I} = \text{alg}\mathcal{L} \).

1. Introduction

Let \( \mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{M}_n \) be a \( C^* \)-algebra of rank \( (p_n) \), where \( (p_n) \) is a strictly increasing sequence of positive integers. Assume that the embeddings of the \( \mathcal{M}_n \) are standard. Define \( \mathcal{T} = \bigcup_{n=1}^{\infty} \mathcal{T}_n \) to be the standard triangular UHF algebra in \( \mathcal{A} \), where \( \mathcal{T}_n \) is the full algebra of upper triangular matrices in \( \mathcal{M}_n \). Let \( \{\pi, \mathcal{H}_\pi\} \) be the tracial representation of \( \mathcal{A} \), and let \( \mathcal{M} \) be the type II₁ hyperfinite factor defined as the weak closure of \( \mathcal{A} \) in \( B(\mathcal{H}_\pi) \). In this paper we study the algebra \( \mathcal{I} = \mathcal{I}^w \), the weak closure of \( \mathcal{I} \) in \( B(\mathcal{H}_\pi) \). We show that \( \mathcal{I} \) is a reflexive, maximal weakly closed triangular algebra in \( \mathcal{M} \). Moreover, we prove that \( \mathcal{I} \) is irreducible relative to \( \mathcal{M} \); i.e., \( \mathcal{M} \cap (\text{lat}\mathcal{I}) = \{0, 1\} \), where \( \text{lat}\mathcal{I} \) is the lattice of all \( \mathcal{I} \)-invariant projections in \( B(\mathcal{H}_\pi) \). We exhibit a strongly closed sublattice \( \mathcal{L} \) of \( \text{lat}\mathcal{I} \) with the property that \( \mathcal{I} = \text{alg}\mathcal{L} \).

The following two theorems are the main results of this paper:

Theorem I. Let \( \mathcal{D}_n \) be the diagonal of the algebra \( \mathcal{M}_n \), and \( \mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{D}_n^w \). Then \( \mathcal{I} \) is a maximal weakly closed triangular algebra in \( \mathcal{M} \) with diagonal \( \mathcal{D} = \mathcal{E}^w \). Moreover, \( \mathcal{I} \) is irreducible relative to \( \mathcal{M} \). Finally, \( \mathcal{I} \) is a reflexive algebra in \( B(\mathcal{H}_\pi) \).

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Theorem II. For each left ideal \( \mathcal{I} \) of \( \mathcal{P} \), let \( P_\mathcal{I} \) be the projection in \( B(\mathcal{H}_x) \) determined by the subspace \( M_\mathcal{I} \), where
\[
M_\mathcal{I} = \bigvee \{ \xi_x | x \in \mathcal{I} \}.
\]
Here \( \xi_x \) is the vector in \( \mathcal{H}_x \) corresponding to the element \( x \in \mathcal{A} \). Then \( P_\mathcal{I} \) is \( \mathcal{I} \)-invariant. Moreover, let \( \mathcal{P} \) be defined as
\[
\mathcal{P} = \{ P_\mathcal{I} | \mathcal{I} \text{ is a left ideal of } \mathcal{P} \},
\]
and define \( \mathcal{L} \) to be the strongly closed subspace lattice in \( B(\mathcal{H}_x) \) generated by \( \mathcal{P} \cup \text{lat} \mathcal{M} \). Then \( \mathcal{L} \subseteq \text{lat} \mathcal{I} \) and \( \mathcal{I} = \text{alg} \mathcal{L} \).

2. Triangularity and maximal properties of \( \mathcal{I} \)

In this section we prove the first part of Theorem I, namely that the algebra \( \mathcal{Y} \) is a maximal weakly closed triangular algebra in \( \mathcal{M} \) with diagonal \( \mathcal{D} \). The main tool developed in this section is presented in Theorem 2.6. In that theorem we construct a family of functions \( \varphi_n : \mathcal{M} \to \mathcal{M}_n \otimes \mathcal{D}^{(n)} \), where \( \mathcal{D}^{(n)} = \bigcup_{k \geq n} M_n \otimes \mathcal{D}_k \) and \( M_n \) is the commutant of \( M_n \) in \( \mathcal{M} \). These functions will allow us to approximate elements of \( \mathcal{M} \) by elements of the subalgebras \( \mathcal{M}_n \otimes \mathcal{D}^{(n)} \). Our construction of the functions \( \varphi_n \) is a direct adaptation of the construction found in [P, Lemma 1.2]. We will need the following three lemmas concerning the factor \( \mathcal{M} : 

Convention. Henceforth, if \( \mathcal{K} \) is a subset of \( \mathcal{M} \), then \( \mathcal{K}^{**}_2 \) will denote the \( \| \cdot \|_2 \)-closure of \( \mathcal{K} \) in \( \mathcal{M} \).

Lemma 2.1 ([MVN, 1.6.1]). Let \( \mathcal{B} \) be a \( * \)-subalgebra of \( \mathcal{M} \). Then \( \mathcal{B}^{**}_2 = \mathcal{B}^w \). Moreover, for each \( x \in \mathcal{B} \), there exists a sequence \( (x_n) \) in \( \mathcal{B} \) such that
\[
\text{sup}_n \|x_n\| < \infty \quad \text{and} \quad x_n = \| \cdot \|_2 \text{-lim}_n x_n.
\]

Lemma 2.2 ([T, III.5.3]). The \( \sigma \)-strong topology on the unit ball of \( \mathcal{M} \) is metrized by the trace norm \( \| \cdot \|_2 \). Moreover, this metric is complete.

Lemma 2.3 ([MVN, 1.3.2]). Let \( x \in \mathcal{M} \), and let \( (x_n) \) be a sequence in \( \mathcal{M} \). Then \( x = s\text{-lim}_n x_n \) if and only if \( x = \| \cdot \|_2 \text{-lim}_n x_n \) and \( \text{sup}_n \|x_n\| < \infty \).

The next result is modeled on [SV, Lemma I.1.6].

Theorem 2.4. Let \( n \) be a positive integer, and let \( \mathcal{M}_n^c \) denote the commutant of \( \mathcal{M}_n \) in \( \mathcal{M} \). Then we have
\[
\mathcal{M}_n^c = \bigcup_{k \geq n} \mathcal{M}_n^c \cap \mathcal{M}_k^{uw}.
\]

Proof. For each \( n \), let \( \{ e^{(n)}_{ij} \} \) be the standard system of matrix units for \( \mathcal{M}_n \). By Lemma 2.1, it suffices to prove that
\[
(*) \quad \mathcal{M}_n^c = \bigcup_{k \geq n} \mathcal{M}_n^c \cap \mathcal{M}_k^{ll_2}.
\]
To prove (*), fix $n$ and define the mapping $Q_n : \mathcal{M} \to \mathcal{M}$ as follows:

$$Q_n(x) = \sum_{i=1}^{p_n} e_{i1}^{(n)} x e_{i1}^{(n)}, \quad \forall x \in \mathcal{M}.$$ 

For $k \geq n$, $Q_n(\mathcal{M}_k) \subseteq \mathcal{M}_n^c \cap \mathcal{M}_k$. Let $x \in \mathcal{M}_n^c$. Then $Q_n(x) = x$. By Lemma 2.1, we have $x = \| \cdot \|_2^2 \lim_{k \geq n} x_k$, where $(x_k)$ is some sequence in $\mathcal{M}$ such that $x_k \in \mathcal{M}_k$. Then $x = \| \cdot \|_2^2 \lim_{k \geq n} Q_n(x_k)$. For $k \geq n$, $Q_n(x_k) \in \mathcal{M}_n^c \cap \mathcal{M}_k$; hence,

$$x \in \bigcup_{k \geq n} \| \cdot \|_2^2 \mathcal{M}_n^c \cap \mathcal{M}_k.$$

This proves (*). \qed

Theorem 2.5. For $n \geq 1$, we have

$$\mathcal{M} = \text{span}\{v x | v \in \mathcal{M}_n, x \in \mathcal{M}_n^c\}.$$

Proof. Use Theorem 2.4 to write

$$\mathcal{M}_n^c = \bigcup_{k \geq n} \mathcal{M}_n^c \cap \mathcal{M}_k,$$

and then imitate the proof of the analogous result in [P, p. 316]. \qed

We are now ready to present the construction of the functions $\varphi_n$.

Theorem 2.6. Let $1 \leq n \leq k$ be positive integers. Let $g_1, \ldots, g_p$ be the minimal projections in $\mathcal{M}_n^c \cap \mathcal{D}_k$. Define the map $\varphi_{nk} : \mathcal{M} \to \mathcal{M}$ by

$$\varphi_{nk}(x) = \sum_{i=1}^{p} g_i x g_i, \quad x \in \mathcal{M}.$$

Then the following conditions hold.

(a) For all $x \in \mathcal{M}$, $\|\varphi_{nk}(x)\| \leq \|x\|$ and $\|\varphi_{nk}(x)\|_2 \leq \|x\|_2$.

(b) For all $x \in \mathcal{M}$, the limit $\varphi_n(x) = \| \cdot \|_2^2 \lim_{k \geq n} \varphi_{nk}(x)$ exists and may be written as

$$\varphi_n(x) = \sum_{i,j=1}^{p_n} e_{ij}^{(n)} d_{ij}, \quad d_{ij} \in \mathcal{D}^{(n)}.$$ 

(c) For all $x \in \mathcal{M}$, $\|\varphi_n(x)\| \leq \|x\|$ and $\varphi_n(x) = \text{s-lim}_{k \geq n} \varphi_{nk}(x)$.

(d) For all $x \in \mathcal{M}$, $x = \text{s-lim}_{k \geq n} \varphi_n(x)$.

Proof. Fix $x \in \mathcal{M}$, and let $k \geq n$. Because the $g_i$ are orthogonal projections with sum 1, we have $\|\varphi_{nk}(x)\| \leq \|x\|$. Now, if $x \in \bigcup_{m=1}^{M} \mathcal{M}_m$, then $\|\varphi_{nk}(x)\|_2 \leq \|x\|_2$. By Lemma 2.1, $\mathcal{M} = \bigcup_{m=1}^{M} \mathcal{M}_m \| \cdot \|_2$; hence, for arbitrary $x \in \mathcal{M}$, we have $\|\varphi_{nk}(x)\| \leq \|x\|_2$. This proves (a).

To prove (b), use Theorem 2.5 to write $x = \sum_{i,j=1}^{p_n} e_{ij}^{(n)} x_{ij}$, $x_{ij} \in \mathcal{M}_n^c$. Now
use Theorem 2.4 and Lemma 2.1 to write \( x_{ij} = \| \cdot \|_2^{-\lim_{k \to \infty} x_{ij}^{(k)}}, \ x_{ij}^{(k)} \in \mathcal{M}_n \cap \mathcal{D}_n, \ \sup_{k \geq n} \| x_{ij}^{(k)} \| < \infty. \) Then \( \varphi_{nk}(x_{ij}^{(k)}) \in \mathcal{M}_n \cap \mathcal{D}_k. \) By (a), \( (\varphi_{nk}(x_{ij}^{(k)}))_{k \geq n} \) is a norm-bounded \( \| \cdot \|_2 \)-Cauchy sequence; hence Lemma 2.2 implies that there exists \( \varphi_n(x_{ij}) \in \mathcal{D}^{(n)} \) such that \( \varphi_n(x_{ij}) = \| \cdot \|_2^{-\lim_{k \to \infty} \varphi_{nk}(x_{ij}^{(k)}).} \) We then have \( \varphi_n(x_{ij}) = \| \cdot \|_2^{-\lim_{k \to \infty} \varphi_{nk}(x_{ij}^{(k)}).} \) Define \( \varphi_n(x) \) by

\[
\varphi_n(x) = \sum_{i, j=1}^{p_n} e_{ij}^{(n)} d_{ij}, \quad d_{ij} = \varphi_n(x_{ij}) \in \mathcal{D}^{(n)}.
\]

Then the limit \( \| \cdot \|_2^{-\lim_{k \to \infty} \varphi_n(x)} \) exists and is equal to \( \varphi_n(x). \) This proves (b).

Condition (c) is a direct consequence of (a), (b), Lemma 2.2, and Lemma 2.3.

To prove (d), note first that, by (b), the sequence \( (\varphi_n(x)) \) is norm bounded. Write \( x = \| \cdot \|_2^{-\lim_{n \to \infty} x_n}, \) where \( x_n \in \mathcal{M}_n. \) Then \( \| x - \varphi_n(x) \|_2 \leq 2 \| x - x_n \|_2. \) Thus \( x = \| \cdot \|_2^{-\lim_{n \to \infty} \varphi_n(x)}. \) It follows from Lemma 2.3 that \( x = \text{s-lim}_n \varphi_n(x), \ x \in \mathcal{M}. \) This proves (d).

The next lemma is a key fact.

**Lemma 2.7.** Let \( n \geq 1, \) and suppose that \( d \in \mathcal{D}^{(n)}. \) If \( i > j \) and \( e_{ij}^{(n)} d \in \mathcal{T}, \) then \( e_{ij}^{(n)} d = 0. \)

**Proof.** Observe that \( e_{ij}^{(n)} d^* = (e_{ij}^{(n)} d) d^* \in \mathcal{T}. \) Write \( e_{ij}^{(n)} d^* = \text{w-lim}_n x_{ij}, \) where \( (x_{ij}) \) is some net in \( \mathcal{S}. \) Then, we have \( \text{w-lim}_n (e_{ij}^{(n)} d^* - x_a) = 0. \)

Define \( a = e_{ij}^{(n)}, \) and let \( \xi_1, \xi_a \) be the vectors in \( \mathcal{M} \) corresponding, respectively, to the elements 1 and \( a. \) We then have \( \lim_{n} \langle (add^* - x_a)\xi_1, \xi_a \rangle = 0. \)

Now, for all \( \alpha, \) we have \( \tau(e_{ij}^{(n)} x_a) = 0; \) therefore, \( \langle (add^* - x_a)\xi_1, \xi_a \rangle = \| e_{ij}^{(n)} d \|^2. \) It follows that \( \| e_{ij}^{(n)} d \|^2 = 0, \) i.e., \( e_{ij}^{(n)} d = 0. \)

We have now developed enough machinery to prove the main results of this section:

**Theorem 2.8.** \( \mathcal{T} \) is a triangular algebra in \( \mathcal{M} \) with diagonal \( \mathcal{T} \cap \mathcal{T}^* = \mathcal{D}. \)

**Proof.** We first show that \( \mathcal{D} \) is a maximal abelian von Neumann subalgebra of \( \mathcal{M}. \) To prove this, let \( \mathcal{F} \) be an abelian von Neumann subalgebra of \( \mathcal{M} \) such that \( \mathcal{D} \subseteq \mathcal{F}. \) Let \( n \geq 1, \) and let \( x \in \mathcal{F} \) be arbitrary. By Theorem 2.6, we have \( \varphi_n(x) = \text{s-lim}_n \varphi_{nk}(x). \) Therefore, \( \phi_n(x) \in \mathcal{F}. \) Therefore, if \( y \in \mathcal{F}, \) then \( \varphi_n(x) \) and \( y \) commute. In particular, \( e_{ij}^{(n)} \varphi_n(x) = \varphi_n(x)e_{ij}^{(n)}, \) for \( 1 \leq i, j \leq p_n. \) By Theorem 2.6, we have

\[
(\ast\ast) \quad \varphi_n(x) = \sum_{i,j=1}^{p_n} e_{ij}^{(n)} d_{ij}, \quad d_{ij} \in \mathcal{D}^{(n)}.
\]

Therefore, if \( i \neq j, \) then \( e_{ij}^{(n)} d_{ij} = 0. \) Thus \( \varphi_n(x) \in \mathcal{D}. \) By Theorem 2.6, we have \( x = \text{s-lim}_n \varphi_n(x), \) consequently, \( x \in \mathcal{D}. \) This proves that \( \mathcal{D} = \mathcal{F}, \) so \( \mathcal{D} \) is a maximal abelian von Neumann subalgebra of \( \mathcal{M}. \)

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To show that $\mathcal{I}$ has diagonal $\mathcal{D}$, let $x \in \mathcal{I} \cap \mathcal{I}^*$, and let $n \geq 1$. Express $\varphi_n(x)$ as in (**). We have $\varphi_n(x) \in \mathcal{I} \cap \mathcal{I}^*$; therefore, $e_{ij}^{(n)} d_{ij} = e_{ii}^{(n)} \varphi_n(x) e_{jj}^{(n)} \in \mathcal{I} \cap \mathcal{I}^*$. Thus, Lemma 2.7 implies that $e_{ij}^{(n)} d_{ij} = 0$ for $i \neq j$. Hence, for all $n$, $\varphi_n(x) \in \mathcal{D}$. By Lemma 2.6, $x = s\text{-}\lim_n \varphi_n(x)$, and hence, $x \in \mathcal{D}$. This proves that $\mathcal{I}$ has diagonal $\mathcal{D}$.

**Theorem 2.9.** $\mathcal{I}$ is a maximal weakly closed triangular algebra in $\mathscr{M}$.

**Proof.** Let $\mathcal{R}$ be a weakly closed triangular algebra in $\mathscr{M}$ such that $\mathcal{I} \subseteq \mathcal{R}$. Then $\mathcal{D} = \mathcal{R} \cap \mathcal{R}^*$. Let $n \geq 1$, and let $x \in \mathcal{R}$. Write $\varphi_n(x)$ as in (**). By (c) of Theorem 2.5, we have $\varphi_n(x) = s\text{-}\lim_{k \geq n} \varphi_{nk}(x)$; hence, because $\varphi_{nk}(x) \in \mathcal{R}$, we have $\varphi_n(x) \in \mathcal{R}$. Let $1 \leq j < i \leq p_n$. Then $e_{ij}^{(n)} d_{ij} = e_{ii}^{(n)} \varphi_n(x) e_{jj}^{(n)} \in \mathcal{R}$. Define $y = e_{ij}^{(n)} d_{ij}$. Then we have $y^* = e_{ji}^{(n)} d_{ij}^* \in \mathcal{I}$; therefore, $y + y^* \in \mathcal{R}$. But then $y + y^* \in \mathcal{R} \cap \mathcal{R}^* = \mathcal{D}$. It follows from Lemma 2.7 that $e_{ij}^{(n)} d_{ij} = 0$. Hence, for all $n$, $\varphi_n(x) \in \mathcal{I}$. Because $x = s\text{-}\lim_n \varphi_n(x)$, we conclude that $x \in \mathcal{I}$. Hence $\mathcal{R} = \mathcal{I}$. $\square$

3. $\mathcal{I}$ is irreducible relative to $\mathscr{M}$

In this section we prove the second part of Theorem I, namely that $\mathcal{I}$ is irreducible relative to $\mathscr{M}$. This proof of this is essentially by reductio ad absurdum.

**Lemma 3.1.** Let $(x_n)$ be a sequence in $\mathcal{I}$ such that $\sup_n \|x_n\| < \infty$. Let $q \in \mathcal{M} \cap (\text{lat}\mathcal{I})$, and suppose that $(q_n)$ is a sequence of projections in $\mathcal{M}$ such that $q = \| \cdot \|_2\text{-}\lim_n q_n$. Then $\lim_n \|q_n x_n q_n\|_2 = 0$.

**Proof.** Verify that for all $n$, $\|q_n x_n q_n\|_2 \leq 3M\|q - q_n\|_2$, where $M = \sup_n \|x_n\|$. $\square$

Straightforward considerations suffice to prove the following lemma; hence we will omit its proof.

**Lemma 3.2.** Let $q \in \mathcal{D}$ be a projection. Then there exists a sequence $(q_n)$ of projections in $\mathcal{M}$ such that $q_n \in \mathcal{D}$ and $q = \| \cdot \|_2\text{-}\lim_n q_n$.

We are now in a position to prove that $\mathcal{I}$ is irreducible relative to $\mathcal{M}$.

**Theorem 3.3.** The algebra $\mathcal{I}$ is irreducible relative to $\mathcal{M}$.

**Proof.** Let $q \in \mathcal{M} \cap (\text{lat}\mathcal{I})$. By Lemma 3.2, there exists a sequence $(q_n)$ of projections in $\mathcal{D}$ such that $q = \| \cdot \|_2\text{-}\lim_n q_n$ and $q_n = \sum_{i=1}^{n} q_i^{(n)} e_{ii}^{(n)}$, with $q_i^{(n)} \in \{0, 1\}$. Let $(N_n)$ and $(L_n)$ be sequences of positive integers defined as follows:

$$N_n = |\{i : 1 \leq i \leq p_n \text{ and } q_i^{(n)} = 0\}|,$$

$$L_n = p_n - N_n.$$

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Then exactly one of the following conditions holds:

(a) \( \liminf_n \frac{N_n}{p_n} = 0 \).
(b) \( \liminf_n \frac{N_n}{p_n} > 0 \).

If (a) holds, then we are done, for then \( \|1 - q\|_2^2 = \lim_n \frac{N_n}{p_n} = 0 \), i.e., \( q = 1 \). Therefore we may assume that (b) holds. We claim that \( q = 0 \). To prove this, assume that \( q \neq 0 \). Then the sequence \( (\frac{N_n}{p_n}) \) is a bounded sequence. Therefore, by (b), there exists an \( \alpha > 0 \) such that some subsequence of \( (\frac{N_n}{p_n}) \) converges to \( \alpha \). Without loss of generality, we may assume that \( \lim_n \frac{N_n}{p_n} = \alpha \). We then have \( \lim_n \frac{L_n}{p_n} = 1 - \alpha \). We cannot have \( \alpha = 1 \); thus, \( 0 < \alpha < 1 \). Now, there exists a sequence \( (X_n) \) of matrices such that for all \( n \), \( X_n \in M_{p_n} \), \( \|X_n\| = 1 \) and

\[
\|X_n\|_2 = \begin{cases} \frac{N_n}{p_n}, & \text{if } N_n \leq L_n; \\ \frac{L_n}{p_n}, & \text{otherwise}. \end{cases}
\]

(d) \( q_n x_n q_n \).

Here \( Q_n = \text{diag}(q_1^{(n)}, \ldots, q_{p_n}^{(n)}) \). Note that the matrix of \( q_n \) with respect to the \( e_{ij}^{(n)} \) is \( Q_n \). Now let \( R_n \in M_{p_n} \) be the matrix defined by \( R_n = W_n \odot X_n \).

Here, \( W_n \) is the \( r_{n+1,n} \times r_{n+1,n} \) matrix having 1’s on the superdiagonal, while all other entries are 0, and \( r_{n+1,n} = p_{n+1}/p_n \). It is clear that \( \|R_n\| = 1 \). Define \( x_n \in \mathcal{M} \) to be the element of \( \mathcal{M} \) whose matrix with respect to the \( e_{ij}^{(n+1)} \) is \( R_n \). Let \( r = r_{n+1,n} \); then, by (d), the matrix of \( q_n x_n q_n \) with respect to the \( e_{ij}^{(n+1)} \) is given by \( (1, \odot Q_n^{\frac{1}{2}}) R_n (1, \odot Q_n^{\frac{1}{2}}) = R_n \). Hence we have \( q_n x_n q_n = x_n \).

It follows that \( \|q_n x_n q_n\|_2 = \|x_n\|_2 = \|R_n\|_2 \). A simple calculation shows that \( \|R_n\|_2^2 = (1 - 1/r_{n+1,n}) \|X_n\|_2^2 \). Because \( p_n < p_{n+1} \), we have \( r_{n+1,n} \geq 2 \), and hence \( (1 - 1/r_{n+1,n}) \geq 1/2 \). Consequently for all \( n \), \( \|q_n x_n q_n\|_2^2 = \|R_n\|_2^2 \geq (1/2) \cdot \|X_n\|_2^2 \). Therefore

\[
\liminf_n \|q_n x_n q_n\|_2 \geq (1/\sqrt{2}) \cdot \liminf_n \|X_n\|_2.
\]

Now, \( \lim_n N_n/p_n = \alpha > 0 \) and \( \lim_n L_n/p_n = 1 - \alpha > 0 \). Therefore condition (c) implies that \( \liminf_n \|X_n\|_2 > 0 \). Hence (e) implies that

\[
\liminf_n \|q_n x_n q_n\|_2 > 0.
\]

Because the matrix \( R_n \) is upper triangular, we see that for all \( n \), \( x_n \in \mathcal{T} \).

Because \( \|x_n\| = \|X_n\| = 1 \), it follows from Lemma 4.1 that \( \lim_n \|q_n x_n q_n\|_2 = 0 \). But this contradicts (e); hence we conclude that \( q = 0 \). This completes the proof of the theorem. \( \Box \)
4. Reflexivity of $\mathcal{F}$

In this final section we prove the last part of Theorem I, namely that $\mathcal{F}$ is a reflexive subalgebra of $B(\mathcal{H}_\infty)$. We conclude the section with a proof of Theorem II.

**Theorem 4.1** ([L, 4.22]). Every weakly closed subalgebra of a finite von Neumann algebra containing m.a.s.a. is reflexive.

**Theorem 4.2.** The algebra $\mathcal{F}$ is reflexive.

**Proof.** Immediate by Theorems 2.8 and 4.1.

**Theorem 4.2.** Let $n \geq 1$, and suppose that $d \in \mathcal{D}^{(n)}$. If $i > j$ and $e_{ij}^{(n)} d \in \mathcal{R}$, then $e_{ij}^{(n)} d = 0$, where $\mathcal{R} = \bigcup_{n=1}^\infty \mathcal{F}^{\parallel z \parallel^2}$. It follows that $\mathcal{R} \subseteq \mathcal{F}$.

**Proof.** Observe that $\langle (\langle \alpha d^* - x \rangle \xi_1, \xi_a) \rangle^2 \leq \parallel e_{ii}^{(n)} \parallel_2^2 \cdot \parallel e_{ii}^{(n)} d d^* - x \parallel_2^2$, for all $i > j, d, x \in \mathcal{A}$, and $a = e_{ij}^{(n)}$. Use this inequality to proceed as in the proof of Theorem 2.7. Now use Theorem 2.6(b) to check that $\mathcal{R} \subseteq \mathcal{F}$. □

**Theorem 4.3.** $\mathcal{F} = \text{alg}\mathcal{L}$.

**Proof.** First, observe that 2.1, 2.6, and 2.7 imply that $\mathcal{L} \subseteq \text{lat}\mathcal{F}$. Hence, to complete the proof, it suffices to show that $\text{alg}\mathcal{L} \subseteq \mathcal{F}$. Let $x \in \text{alg}\mathcal{L}$. Then $x \in \text{alg}\text{lat}\mathcal{M} = \mathcal{M}$. We claim that for all $y \in \mathcal{F}$, $xy \in \mathcal{F}$; hence, with $y = 1$, we get $x \in \mathcal{F}$. To prove the claim, write $x = \parallel \cdot \parallel_2\text{-lim}_n x_n$, where $x_n \in \mathcal{M}$ and $\sup_n \parallel x_n \parallel < \infty$. Let $y \in \mathcal{F}$. Then $x \xi_y \in M_{\mathcal{F}}$; therefore, there exists a sequence $(\gamma_n)$ in $\mathcal{F}$ such that $x \xi_y = \parallel \cdot \parallel_2\text{-lim}_n \xi_{\gamma_n}$. We then have $\parallel x_n \xi_y - \xi_{\gamma_n} \parallel \leq \parallel x_n \xi_y - x \xi_y \parallel + \parallel x \xi_y - \xi_y \parallel$. Because $x = \text{s-lim}_n x_n$, it follows that $\parallel x_n y - y \parallel_2^2 = \parallel \xi_{(x_n y - y)} \parallel^2 = \parallel x_n \xi_y - \xi_y \parallel^2 \rightarrow 0$. Hence we have $\lim_n \parallel xy - y \parallel_2 = 0$. Therefore, $xy = \parallel \cdot \parallel_2\text{-lim}_n y_n$. Theorem 4.2 then gives $xy \in \mathcal{R}$, i.e., $xy \in \mathcal{F}$. □

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**References**


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