UNIFORM $L^2$-WEIGHTED SOBOLEV INEQUALITIES

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Abstract. We prove the inequality (1) for weights $w$ in a class which contains
the class $J_p$, $p > (n - 1)/2$, introduced by C. Fefferman and D. H. Phong in
studying eigenvalues of Schrödinger operators. In our case, $C$ is independent
of the lower order terms of $P$. As a consequence we prove unique continuation
theorem for solutions of $\Delta + V$, $V$ in the same class.

INTRODUCTION

We give conditions on weight functions $w$ so that an inequality

(1) $\|u\|_{L^2(w)} \leq C\|P(D)u\|_{L^2(w^{-1})}$

holds for the constant coefficient operator in $\mathbb{R}^n$

$P(D) = \Delta + \sum a_i \partial / \partial x_i + b$,

where $n \geq 3$ and the constant $C$ is independent of the lower order terms $a_i$, $b$ in $C$.

The weight $w$ must satisfy two conditions. The first is a one-dimensional
A$_2$ Muckenhoupt's requirement, defined at the beginning of §1, which depends
only on the direction of the vector $a_j$. The second is an estimate,

$$\left( \frac{1}{|B_r|} \int_{B_r} w^\alpha(x) \, dx \right) \left( \frac{1}{|B_r|} \int_{B_r} w^{-\alpha}(x) \, dx \right)^{-1} \leq C r^{-4\alpha},$$

where $B_r$ denotes a ball of radius $r$ and $C$ is independent of $B_r$.

A similar, weaker condition for the particular case of the Laplace operator
was given in [CW] (see for instance (1, 6) in this reference).

Also, an inequality like (1) has been proved in [KRS] for $L^p$ norms; more
precisely

(0) $\|u\|_{L^p} \leq C\|P(D)u\|_{L^p}$,

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where $p$ and $p'$ satisfy the duality condition and the Sobolev gap $1/p - 1/p' = 2/n$.

The ingredients to prove (1), as in [KRS], are a similar uniform inequality
\begin{equation}
\|u\|_{L^2(w)} \leq C(\|\Delta + z\|u\|_{L^2(w^{-1})}),
\end{equation}
and a weighted restriction theorem for the Fourier transform which we state independently, since it requires a weaker hypothesis.

As a consequence of (1), we prove a version of a result due to Chanillo and Sawyer [CS] on unique continuation of solutions of Schrödinger equations. More precisely, we prove that a solution $u$ of the inequality
\[ |\Delta u(x)| \leq |V(x)u(x)|, \]
which is zero in an open set, must vanish everywhere. We assume some minimal conditions on $u$, and assume $V$ to be in the Fefferman-Phong class
\[ J^\text{loc}_{p,F} = \left\{ V \in L^p_{\text{loc}}, \text{ such that } \limsup_{r \to 0} \sup_{x \in K} r^{-n} \int_{B_r(x)} V^p \leq F, \right\}, \]
for $p > (n-1)/2$ and $F$ some constant depending only on dimension.

Chanillo and Sawyer (see [K]) obtained a stronger unique continuation for $V$ in the Fefferman-Phong class for $F = 0$ and the same $p$’s; their proof relies on an $L^2 \to L^2$ estimate due to Jerison and Kenig very difficult to prove. We take from them the idea of substituting weighted $L^2$ inequalities for $L^p$ inequalities and also the use of maximal functions in the unique continuation context.

Our proof of (1), and hence a new proof of unique continuation property for potential in the class $J^\text{loc}_{p,F}$, relies on restriction theorems for the Fourier transform, stationary phase methods, and real interpolation.

1. Statement of results and consequences

We start with some definitions.

Let $w$ be a nonnegative function. We say that $w$ is in $D_\beta$, $\beta < 0$, if there exists a constant $C > 0$ such that for any ball $B_t$ of radius $t$,
\[ \left( t^{-n} \int_{B_t} w(x) \, dx \right) \left( t^{-n} \int_{B_t} w(x)^{-1} \right)^{-1} \leq Ct^\beta, \]
we denote by $|||w|||^2$ the infimum of such constants $C$.

Let $y$ in $R^n$, and let us write for any $y$ in $R^n$, $y = ty + y'$, where $y'$ is in some hyperplane $\Pi$ of $R^n$. We will say that $w$ is in $A_p(y)$ if the function
\[ w_{y'} : R \to R^+, \text{ defined by } w_{y'}(t) = w(y') \text{ is an } A_p \text{ weight in } R \text{ for almost every } y' \in \Pi, \text{ with } A_p \text{ norm uniformly bounded.} \]
Theorem 1. Let \( P(D) = \Delta + \sum a_j \partial / \partial x_j + b, a_j, b \in \mathbb{C} \). Then there exists a direction \( \gamma \) in \( \mathbb{R}^n \), depending only on the direction of the vector \((\text{Re} a_j)\), such that if \( w \) is in \( A_2(\gamma) \) and \( w^\alpha \) is in \( D_\beta \) for some \( \alpha > (n-1)/2 \), \( \beta = -4\alpha \), the following inequality (1)

\[
\|u\|_{L^2(v)} \leq C \|\|w^\alpha\|^{1/\alpha}\|P(D)u\|_{L^2(v^{-1})}^{1/\alpha}
\]

holds with an absolute constant \( C \) depending only on the \( A_2(\gamma) \) constant of \( w \).

The proof of the above theorem is based on the following

Theorem 2 (Weighted restriction theorem for the Fourier transform). Let \( d\sigma \) be the uniform measure on the sphere \( S^{n-1} \) in \( \mathbb{R}^n \) and \( (d\sigma)^{-1} \) its Fourier transform; let \( w \) be a doubling weight \( w > 0 \), i.e. there exists a constant \( C' \) such that

\[
\int_{B_r} w \leq C' \int_{B_r} w.
\]

Then for \( w^\alpha \in D_\beta \) for \( 1 < \alpha < (1-n-\beta)/2 \) and \( n \geq 3 \), there exists a constant \( C \) independent of \( \|\|\|u\|\| \) such that

\[
\|d\sigma)^{-1} * f\|_{L^2(v)} \leq C \|\|w^\alpha\|^{1/\alpha}\|f\|_{L^2(v)}
\]

for every \( f \in C_0^\infty \).

Theorem 3 (Weighted \( L^2 \) restriction theorem for the resolvent). Let \( w \) be as in Theorem 2, then there exists a constant \( C \) such that

\[
\|u\|_{L^2(v)} \leq C \|\|w^\alpha\|^{1/\alpha}\|z|^{-1-(\beta/4\alpha)}\|(-\Delta + z)u\|_{L^2(v^{-1})}^{1/\alpha}
\]

for any \( u \in C_0^\infty \) and \( z \) in \( \mathbb{C} \).

The following method gives weights which satisfy the hypothesis of Theorem 1, with \( A_2(\gamma) \) constant depending only on the dimension.

Take \( \gamma_1, \gamma_2, \ldots, \gamma_n \), an orthogonal basis of \( \mathbb{R}^n \) and define the strong maximal function

\[
Mf(x) = \sup_R |R|^{-1} \int_R |f(y)| dy,
\]

where the sup is taken over all the rectangles with edges in the directions \( \gamma_1 = \gamma, \gamma_2, \ldots, \gamma_n \), which contain \( x \).

Let us define \( M_\alpha f = (M(f^\alpha))^{1/\alpha} \). Then we have the following lemma due to Chanillo and Sawyer:

Lemma 1. Let \( f \) be in \( J_{p,F} \) for \( p > (n-1)/2 \), and \( 1 < \alpha < p \); then there exists a constant \( C_\alpha \), depending only on the dimension and \( \alpha \), such that \( M_\alpha f \) is in \( J_{p,F'} \), for \( F' = C_\alpha F \).

Now we are in position to construct the appropriate weights. Following [R], let us take

\[
Sf = \sum_{j=0}^\infty (2C_\alpha)^{-j} \|M_\alpha f\|.
\]
Then if $V \in J^{\text{loc}}_{p,F}$, define
\begin{equation}
(4) \quad w(x) = S(V \chi_{B_R}).
\end{equation}

The following properties hold:
(a) $M_{\alpha} w(x) \leq C w(x)$ for some $C \geq 0$; this is a $A^*_{\alpha}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ condition for $w^\alpha$, hence $w$ satisfies the same property.
(b) $w$ is in $J_{p,F}$, if $\alpha < p$, $p > (n - 1)/2$;
(c) $w^\alpha$ is in $D_\beta$ for $\beta = -4\alpha$ and $|||w^\alpha|||^{1/\alpha} \leq C_{\alpha,F}$;
(d) $w$ is in $A_2(\gamma)$.

Conditions (a) and (b) are consequences of the results in [GR, page 433] and Lemma 1.
Condition (c) is a consequence of Jensen’s inequality.
Condition (d) follows from Theorem 6.2 in [GR].

As a consequence of Theorem 1, we prove the following unique continuation property:

**Corollary.** There exists $F \geq 0$, depending only on the dimension, such that for any solution of the differential inequality
\begin{equation}
(5) \quad |\Delta u(x)| \leq V(x)|u(x)|,
\end{equation}
where $V$ is in $J^{\text{loc}}_{p,F}$ for some $p > (n - 1)/2$ and $u$ is in $H^{1,2}_{\text{loc}}$, if $u$ is zero in an open set, $u$ must be zero everywhere.

$H^{1,2}_{\text{loc}}$ denotes the Sobolev space of functions in $L^2_{\text{loc}}$ with derivatives in $L^2_{\text{loc}}$.

**Proof of the corollary.** When $w$ satisfies the hypothesis of Theorem 1, the following Carleman inequality holds, with constant $C$ independent of $v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$:
\begin{equation}
(6) \quad \|e^{\lambda v \cdot x} u\|_{L^2(w)} \leq C \|w^\alpha\|^{1/\alpha} \|e^{\lambda v \cdot x} \Delta u\|_{L^2(\mathbb{R}^n)}.
\end{equation}

In fact (6) can be reduced, by changing $v = e^{\lambda v \cdot x} u$ to
\begin{equation}
(6') \quad \|v\|_{L^2(w)} \leq C \|w^\alpha\|^{1/\alpha} \|P(D)v\|_{L^2(\mathbb{R}^n)},
\end{equation}
where $P(D) = \sum |D_j + i\lambda v_j|^2$, $v = (v_j)_{j=1,\ldots,n}$, which is a particular case of (1).

By a reflection argument (see [KRS]) we may assume that $u = 0$ out of a compact set, and hence (by translation and dilation) to the case $u = 0$ out of $B_1 = \{x \in \mathbb{R}^n, \ x_1^2 + \cdots + x_{n-1}^2 + (x_n + 1)^2 \leq 1\}$; it suffices to prove that $u = 0$ in a neighborhood of the origin.

The above transformations preserve the spaces $J_p$. Also, by adding to $V$ the function $e|x|^{-2}$, we obtain a new potential in (5), which is in $J^{\text{loc}}_{p,F+\varepsilon}$ and is bounded below in compact sets.
We still call $V$ the resulting potential of applying rotations, inversions, and the above perturbation to $V$ in (5).

Take now $w$ as (4) after Lemma 1, with $\gamma$ given by Theorem 1 when $a_j = 2\nu_j$. From Conditions (a)–(d), $w$ satisfies the hypothesis of this theorem.

Let us choose $\eta \in C^\infty_0(B_{B\delta})$, $\eta = 1$ in $|x| \leq \delta/2$, $\delta > 0$ to be fixed, then (6) with $\nu = (0, \ldots, 0, 1)$ gives for $\delta$ small

$$||e^{\lambda x_n} g||_{L^2(w)} \leq 2CF ||e^{\lambda x_n} \Delta g||_{L^2(w^{-1})},$$

where $g = u\eta$,

$$\leq 2CF ||e^{\lambda x_n} uV||_{L^2(w^{-1}dx, B_{B\delta/2})} + C' ||e^{\lambda x_n} g||_{L^2(w^{-1}dx, B_{B\delta/2})}$$

but

$$||e^{\lambda x_n} uV||_{L^2(w^{-1}dx, B_{B\delta/2})} = \left( \int_{B_{B\delta/2}} |e^{\lambda x_n} u(x)|^2 V^2(x) w^{-1}(x) \, dx \right)^{1/2}$$

$$\leq \left( \int_{B_{B\delta/2}} |e^{\lambda x_n} u(x)|^2 w(x) \, dx \right)^{1/2}$$

since $V(x) \leq S(V_{x_B}) = w(x)$ for $\delta$ small enough.

Assuming $2CF \leq 1/4$, we obtain

$$||e^{\lambda x_n} g||_{L^2(wdx, B_{B\delta/2})} \leq 2C' ||e^{\lambda x_n} \Delta g||_{L^2(w^{-1}dx, B_{B\delta/2})}$$

uniformly in $\lambda$; then there exists $\zeta > 0$ such that $g = 0$ on $x_n > -\zeta$, if we prove that the right-hand side is finite.

$$||\Delta g||_{L^2(w^{-1}dx, B_{B\delta/2})}$$

$$\leq ||(\Delta \eta) u||_{L^2(w^{-1}dx, B_{B\delta})} + ||\nabla \eta \cdot \nabla u||_{L^2(w^{-1}, B_{B\delta})} + ||\eta \Delta u||_{L^2(w^{-1}, B_{B\delta})}$$

$$\leq C ||u||_{L^2(dx, B_{B\delta})} + C ||\nabla u||_{L^2(dx, B_{B\delta})} + C ||V u||_{L^2(w^{-1}dx, B_{B\delta})}$$

since $w^{-1}$ is bounded in $B_{B\delta}$, and $u$ is in $H^{1,2}_{loc}$. To bound the last term consider that

$$\int |V u|^2 w^{-1} \, dx \leq \int |u|^2 V \, dx \leq C \int |\nabla u|^2 \, dx,$$

since $V$ is in the Fefferman-Phong class with $p \geq 1$, and Sobolev estimates hold [FP].

3. The proofs

Proof of Theorem 1. As in [KRS], by rotation, density, and replacing $u$ by $ue^{-ixc}$ for some $c$ in $R^n$, we may reduce to the case

$$P(D) = \Delta + \sigma + \tau(\partial/\partial x_n + ie).$$

We have to prove (1) with $C$ independent of $\sigma$, $\tau$, $e$ in $R\{0\}$. 

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Take
\[ m(\xi) = (-|\xi|^2 + \sigma + i\tau(\xi_n + \epsilon))^{-1}; \]
we have to prove
\[ |(m(\xi)\hat{f}(\xi))\widehat{\nabla}||_{L^2(w)} \leq C||w^\alpha||^{1/\alpha}||f||_{L^2(w^{-1})}. \]

By inserting a cutoff function \( \chi \) such that \( \chi(t) = 1 \) on \( 1 < |t| < 2 \), we may write
\[
(m(\xi)\hat{f}(\xi))\widehat{\nabla}(x) = \left( \sum_k m_k(\xi)\hat{f}(\xi) \right)\widehat{\nabla}(x),
\]
where \( m_k(\xi) = \chi_k(\xi)m(\xi) \) and \( \chi_k(\xi) = \chi(2^k(\xi_n + \epsilon)) \). Then (7) is bounded by
\[
\left| \sum_k \int |\{m_k\hat{f}\widehat{\nabla}(x)|^2w(x)\,dx \right|^{1/2} \leq C||w^\alpha||^{1/\alpha} \left( \sum_k \int |\{\chi_k(\cdot)\hat{f}(\cdot)\widehat{\nabla}(x)|^2w^{-1}(x) \right)^{1/2}
\]
and then, by the above argument, bounded by
\[ C||w^\alpha||^{1/\alpha}||f||_{L^2(w^{-1})}. \]

Only (8) remains to be proved; it may be reduced to the boundedness of the Fourier multiplier
\[ m_k^*(\xi) = \chi_k(\xi)(-|\xi|^2 + \sigma + i\tau 2^{-k})^{-1}, \]
which is bounded by Theorem 3 with \( z = \sigma + i\tau 2^{-k} \).

Hence only the difference \( m_k - m_k^* \) has to be bounded as a Fourier multiplier; it is given by
\[
(Tf)\widehat{\nabla}(\xi) = \chi_k(\xi_n)f\widehat{\nabla}(\xi)i\tau(\xi_n + \epsilon - 2^{-k})
\times (-|\xi|^2 + \sigma + i\tau(\xi_n + \epsilon))^{-1}(-|\xi|^2 + \sigma + i\tau 2^{-k})^{-1}.
\]

Take polar coordinates \( \xi = \rho \xi' \), by Minkowski’s inequality
\[
\int_0^\infty \left\| \int_{S^{n-1}} (Tf)(\rho \xi')e^{ip\xi'\cdot x} \,d\sigma(\xi') \right\|_{L^2(w(x)) \,dx} \rho^n \,d\rho = \int_0^\infty \|(d\sigma)\cdot\xi(m_{\rho}(\xi_n)\hat{f}(\rho \xi))\widehat{\nabla}(\rho x)\|_{L^2(w)\rho \,d\rho} \rho^{n-1} \,d\rho,
\]
where
\[
m_{\rho}(\xi_n) = \chi_k(\rho \xi_n)i\tau(\rho \xi_n + \epsilon - 2^{-k})
\times (-|\rho|^2 + \sigma + i\tau(\rho \xi_n + \epsilon))^{-1}(-|\rho|^2 + \sigma + i\tau 2^{-k})^{-1} = \int_0^\infty \|(d\sigma)\cdot\xi(m_{\rho}(\xi_n)\hat{f}(\rho \xi))\widehat{\nabla}(y)\|_{L^2(w(y/\rho)) \,dy} \rho^{n/2-1} \,d\rho \leq \int_0^\infty \|(m_{\rho}(\xi_n)\hat{f}(\rho \xi))\widehat{\nabla}(y)\|_{L^2(w(y/\rho)^{-1}) \,dy} \rho^{n/2-1-\beta/2\alpha} \,d\rho ||w^\alpha||^{1/\alpha},
\]
by Theorem 2, and
\[
|||w^\alpha(x/\rho)|||^2 = \rho^{-\beta}|||w^\alpha(x)|||^2,
\]
\[
= \int_0^\infty \{\rho^{-\alpha} m_\rho(\rho^{-1} \xi_n) \hat{f}(\xi)\} \hat{(y/\rho)}|||L^2(w(y/\rho)^{-1})dy\rho^{n/2-1-\beta/2\alpha}|||w^\alpha|||^{1/\alpha}.
\]
But since
\[
m_\rho(\rho^{-1} \xi_n) = \chi_k(\xi_n) i\tau(\xi_n + \epsilon - 2^{-k})| - \rho^2 + \sigma + i\tau(\xi_n + \epsilon)|^{-1} - \rho^2 + \sigma + i\tau 2^{-k}||^2
\]
is an \(L^2(w^{-1}) \rightarrow L^2(w^{-1})\) operator with norm
\[
C \frac{\tau 2^{-k}}{(\rho^2 - \sigma)^2 + (\tau 2^{-k})^2},
\]
where \(C\) depends only on the \(A_2(\gamma)\) constant of \(w^{-1}\), we have
\[
\|Tf\|_{L^2(w)} \leq C |||w^\alpha|||^1 \int_0^\infty \frac{\rho^{-1-\beta/2\alpha} \tau 2^{-k}}{(\rho^2 - \sigma)^2 + (\tau 2^{-k})^2} \|f\|_{L^2(w^{-1})} \rho^{n/2-1-\beta/2\alpha} \|w^\alpha\|^{1/\alpha},
\]
with \(\lambda = \tau 2^{-k}/\sigma\); if we take \(-\beta/4\alpha - 1 = 0\), the integral is uniformly bounded with respect to \(\lambda\) and the theorem is proved.

\textbf{Proof of Theorem 2.} It is known that
\[
(d\sigma)^\wedge = |x|^{-(n/2)+1} J_{(n/2)-1}(|x|), \quad \text{where } J_t \text{ denotes the Bessel function of order } t.
\]
If \(|x| \geq 1\), \(J_{(n/2)-1}(|x|)\) is asymptotically like \(e^{i|x|} |x|^{1/2}\).

Let us take a cutoff function \(\psi(|x|)\) such that \(\text{supp } \psi\) is contained in \([1, 2]\) and
\[
\sum_{k=1}^\infty \psi(2^{-k}s) = 1 \quad \text{for } |s| > 1;
\]
write \(d\sigma = \sum_{k=0}^\infty F_k(x) = \sum_{k=0}^\infty \psi_k(x) d\sigma(x)\), where \(F_k(x) = \psi(2^{-k} |x|)\).

Take \(T_k\) the operator of convolution with \(F_k\). We are going to estimate its \(L^2(w^{-\theta}) \rightarrow L^2(w^{\theta})\) mapping norm:

For \(\theta = 0\) we use P. Tomas’s estimate (see [T]),
\[
\|T_k f\|_2 \leq C 2^k \|f\|_2.
\]

For \(\theta = \alpha > 1\),
\[
|F_k(x)| \leq C |\psi(2^{-k} |x|)| e^{i|x|} |x|^{-(n-1)/2} \leq C 2^{-k(n-1)/2} \chi_{\{|x| < 2^{k+1} \}}(|x|).
\]
Hence
\[ |(T_k f)(x)| \leq C 2^{-k(n-1)/2} \int_{|x-y| \leq 2^k} |f(y)| \, dy \]
\[ \leq 2^{-k(n-1)/2} M^{2k}(f(x)) 2^{nk} , \]
where \( M_t f(x) = \sup_{|B| > r} \int_B |f(x-y)| \, dy \) is the truncated Hardy-Littlewood maximal function.

We invoke the following lemma (see, for instance, [GR]):

**Lemma 2.** Let \( V \) be a measure in \( \mathbb{R}^{n+1} \), and \( u \) a doubling measure in \( \mathbb{R}^n \), such that for any cube \( I_h \) of side length \( h > 0 \), there exists a constant \( K > 0 \) such that
\[ V(I_h \times [0, h]) \leq Ku(I_h) ; \]
then
\[ \left| \int_{\mathbb{R}^{n+1}} |M_{s} f(x)|^2 \, dV(x, s) \right|^{1/2} \leq K^{1/2} \left( \int_{\mathbb{R}^n} |f(x)|^2 \, du \right)^{1/2} . \]

We take \( dV(x, t) = w^\alpha \otimes \delta_t \), where \( \delta_t \) is the singular measure on \( s = t \) in \( \mathbb{R}^n \), and \( du(x) = w^{-\alpha}(x) \, dx \); in this case the Carleson constant \( K \) is bounded by \( |||w^\alpha|||^2 t^\beta \), in fact:
\[ w^\alpha \otimes \delta_t(I_h \times [0, h]) = \delta_t([0, h]) w^\alpha(I_h) \]
\[ = \chi_{t \geq h}(h) w^\alpha(I_h) = \int_{I_h} w^\alpha(x) \, dx \chi_{t, \infty}(h) \]
\[ \leq \int_{I_h} w^{-\alpha}(x) \, dx |||w^\alpha|||^2 h^\beta \chi_{t, \infty}(h) \]
\[ \leq w^{-\alpha}(I_h) |||w^\alpha|||^2 t^\beta (\beta < 0) . \]

Hence
\[ \left( \int |M_{s} f(x)|^2 \, dw^\alpha(x) \right)^{1/2} \leq |||w^\alpha||| t^{\beta/2} \left( \int |f(x)|^2 w^{-\alpha}(x) \, dx \right)^{1/2} \]
and
\[ \|T_k \|_{L^2(w^{-1}) \rightarrow L^2(w)} \leq C 2^{k(1-1/\alpha) 2^{-(k(n-1)/2+nk+\hat{k}/2)1/\alpha}} |||w^\alpha|||^{1/\alpha} . \]

We apply interpolation and
\[ \|T_k \|_{L^2(w^{-1}) \rightarrow L^2(w)} \leq C 2^{k(1-1/\alpha) 2^{-(k(n-1)/2+nk+\hat{k}/2)1/\alpha}} |||w^\alpha|||^{1/\alpha} \]
\[ = C 2^{k(1-(n-\beta)/(2\alpha))} |||w^\alpha|||^{1/\alpha} , \]
and the sum is convergent for \( 2 < 2\alpha < (1 - n - \beta) \).

**Proof of Theorem 3.** We follow along the lines of the proof of Theorem 2.

Take \( m(\xi) = (|\xi|^2 + z)^{-1} \) as a Fourier multiplier; we may assume \( \text{Im} \, z \neq 0 \) by a density argument.
The Fourier transform of \( m \) is given by
\[
K(x) = C \left( \frac{z}{|x|^2} \right)^{1/2(n/2-1)} K_{n/2-1}((z|x|^2)^{1/2}),
\]
where \( K_{n/2-1} \) is a modified Bessel function (see [KRS] and [GS]) whose behavior is
\[
|K_{n/2-1}(t)| \leq C |t|^{-n/2+1}, \quad \text{for } |t| \leq 1, \quad \text{Re } t > 0,
\]
\[
K_{n/2-1}(t) = b(t)t^{-1/2}e^{-t}, \quad \text{for } |t| > 1, \quad \text{Re } t > 0,
\]
where \( b(t) \) satisfies
\[
|\left(\frac{d}{d\rho}\right)^j b(\rho t/|t|)| \leq C_j |\rho|^{-j}.
\]
We take a determination of \((z|x|^2)^{1/2}\) with positive real part. Now,
\[
|K(x)| \leq C |x|^{-n+2} \quad \text{for } |z|^{1/2}|x| \leq 1,
\]
with \( C \) independent of \( z \), and we may consider the convolution operator with \( \psi_k(r|x|)K(x) \). One can see that the worst case occurs when \( \text{Im } z \) is small. In this case the operator behaves like the restriction operator dilated by \( |z| \). This is the idea in the following calculation:

(a) \( T_k: L^2 \to L^2 \) with norm \( 2^k/|z| \). In fact, if we denote \( s = |x| \) and \( z^{1/2} = r \cos \alpha + ir \sin \alpha \), the multiplier is
\[
\begin{align*}
&z^{(n-2)/2} \int_{R^n} \psi_k(r|x|)K_{(n/2)-1}(z^{1/2}|x|)|z|x|^2|^{-(n-2)/4} e^{ix\xi} \, dx \\
&= z^{(n-2)/2} \int_{2^{k-1} < rs < 2^k} \psi_k(rs)b(z^{1/2}s)(zs^2)^{-(n-1)/4} e^{-irs \sin \alpha - rs \cos \alpha} \\
&\quad \cdot \int_{s^{-1}} e^{is\omega \xi} d\omega s^{-1} \, ds \\
&= z^{(n-2)/2} \int_{2^{k-1} < rs < 2^k} \psi_k(rs)b(z^{1/2}s)(zs^2)^{-(n-1)/4} e^{-irs \sin \alpha - rs \cos \alpha} \\
&\quad \cdot a(s|\xi|)(s|\xi|)^{-(n-1)/2} e^{is|\xi|} s^{-1} \, ds,
\end{align*}
\]
where \( a \) is a function with the same type of estimates as \( b \).

Since \( \cos \alpha > 0 \), if \( \cos \alpha > \eta > 0 \) the above integral is bounded by \( C \exp(-2^k \eta r) \). Hence the worst case is when \( 0 < \cos \alpha < \eta \); we use the stationary phase method and obtain the bound
\[
|z|^{(n-2)/2} |r \sin \alpha|^{-(n-1)/2} 2^k r^{-1} \approx C 2^k r^{-2}.
\]

(b) \( T_k: L^2(w^{-\alpha}) \to L^2(w^\alpha) \) with norm \( r^{2-(\beta/2)} 2^{k(n+1+\beta)/2} |||w^\alpha||| \), by a repetition of the argument in Theorem 2.

Using interpolation we obtain the range of convergence \( 2 < 2\alpha < (1 - \beta - n) \) and the operator norm \( L^2(w^{-1}) \to L^2(w) \) of \( T_k \)
\[
Cr^{2-(\beta/2\alpha)} |||w^\alpha|||^{1/\alpha}.
\]
This is independent of \( r \) for \( \beta = -4\alpha \).
Proof of Lemma 1. We may assume, without any restriction, that the $\gamma$'s are in the coordinates' directions.

Define

$$M_{j, \alpha} f(x) = \left( \sup_{a < x_j < b} \frac{1}{b - a} \int_{[a, b]} |f(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)|^\alpha \, dt \right)^{1/\alpha}.$$ 

We see that $M_{\alpha} f(x) \leq M_{1, \alpha} \cdots M_{n, \alpha} f(x)$; hence if we prove that for any $j = 1, \ldots, n$, there exists a constant $C'_\alpha$ such that for any $g \in J_{p, F}$, $M_{j, \alpha} g \in J_{p, F'}$, for $F' = C'_\alpha F$, a recurrent application of this fact would prove our statement with $C = C'_\alpha^n$.

Let $Q$ be a cube centered at $z = (z_1, \ldots, z_n)$ of side length $\delta$; in order to evaluate the average of $M_{1, \alpha}$ in $Q$, we may assume that $f$ is supported on the strip

$$S = \{x \in \mathbb{R}^n, \text{ there exists } t \text{ with } (t, x_2, \ldots, x_n) \text{ in } Q\}.$$ 

Decompose $S$ in pair of rectangles $R_k \cup R_{-k}$ in $Q$, where

$$R_k \cup R_{-k} = \{y \in \mathbb{R}^n, |y_j - z_j| < \delta, j = 2, \ldots, n, 2^k \delta < |y_1 - y_n| < 2^{k+1} \delta\}, \quad k = 1, 2, \ldots.$$ 

Let us denote

$$f_k = f x_{R_k}, \quad f_0 = f x_Q, \quad f_{-k} = f x_{R_{-k}}.$$ 

Then

$$\left( \delta^{-n} \int_Q (M_{1, \alpha}(f)^p)^{1/p} \right)^{1/p} \leq \left( \sum_{k = -\infty}^{\infty} \delta^{-n} \int_Q (M_{1, \alpha} f_k)^p \right)^{1/p}.$$ 

For $k = 1, 2, \ldots$, and $x$ in $Q$

$$M_{1, \alpha} f_k(x) \leq \left( \frac{1}{2^{k+1} \delta} \int_{|z_1 - t| \leq 2^{k+1} \delta} |f_k|^\alpha \, dt \right)^{1/\alpha};$$ 

hence since $\alpha < p$,

$$\delta^2 \left( \delta^{-n} \int_Q |M_{1, \alpha} f_k|^p \right)^{1/p} \leq \delta^2 \left( \frac{1}{|R_k|} \int_{R_k} |f|^p \, dx \right)^{1/p} \leq \delta^2 \left( 2^{k(n-1)} |Q_k|^{-1} \int_{Q_k} |f|^p \right)^{1/p},$$

where $Q_k$ is a cube of side length $2^{k+1} \delta$. 

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The above expression is bounded by $2^{k(n-1)/p-2k}F$. For $f_0$, since $p/\alpha > 1$ and $M_1$ is bounded in $L^{p/\alpha}$, we have

$$\delta^2 \left( |Q|^{-1} \int_Q |M_{1,\alpha} f_0|^p \right)^{1/p} \leq C \delta^2 |Q|^{-1/p} \left( \int_Q |M_1 f_0|^{p/\alpha} \right)^{1/p} \leq C \delta^2 \left( 1/|Q| \int_Q |f|^p \right)^{1/p} \leq C \alpha F.$$

For $k = -1, -2, \ldots$, we obtain similar bounds (with $|k|$ in the exponent). The sum is convergent if $(n-1)/2 < p$.

**Remark and open questions.** The condition $u$ in $H^{1,2}$ in the corollary is chosen to avoid a more complicated condition, which states that $\nabla u \in L^2(\omega^{-1})$ and $u \in L^2(\omega)$, for the $A_1$ weights constructed in the proof. In [KRS] the similar requirement is $u \in H^{2,p}$, $p = 2n/(n+2)$.

In [KRS] the inequality (0) is proved for the operator $\partial^2/\partial t^2 - \Delta_n$ lower order; our proof involves the geometry of the level sets of the kernel; in the Klein-Gordon case this geometry is more complicated. We wonder if there is an appropriate class of potentials $V$ adapted to this case.

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**REFERENCES**


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