PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 2, June 1991

# $\alpha$-INVARIANT AND $S^1$ ACTIONS

KAORU ONO

(Communicated by Frederick R. Cohen)

ABSTRACT. If a closed spin manifold admits an $S^1$ action of odd type, then its $\alpha$-invariant vanishes.

## INTRODUCTION

A spin structure on an $m$-dimensional oriented manifold $M$ is a principal Spin($m$) bundle $P$ on $M$, which is a fiberwise double covering of the oriented orthonormal frame bundle of $M$. The $\alpha$-invariant is a spin cobordism invariant with values in $KO^*(pt)$ and coincides with the $\hat{A}$-genus in case of $4k$-dimension [A-B-S], [A-S V], [H], [M]. The $\hat{A}$-genus of spin manifolds is an obstruction to metrics of positive scalar curvature [L] and an obstruction to nontrivial $S^1$ actions [A-H]. It is known that, if $M$ carries a metric of positive scalar curvature, the $\alpha$-invariant $\alpha(M)$ vanishes [H]. On the other hand the $\alpha$-invariant does not necessarily vanish for spin manifolds admitting nontrivial $S^1$ actions. For example, the circle $S^1$ with the product spin structure has nonvanishing $\alpha$-invariant and admits an $S^1$ action of even type. Here an $S^1$ action on $M$ is called of even (resp. odd) type, if the $S^1$ action can (resp. cannot) be lifted to the spin structure $P$. ($S^1$ has another spin structure which is induced from the spin structure of the two-dimensional disk $D^2$. The $\alpha$-invariant of this spin manifold is zero, and the $S^1$ action is of odd type.) Since the $\alpha$-invariant is an algebraic homomorphism from the spin cobordism ring to $KO^*(pt)$, product manifolds of $S^1$ with the product spin structure and 8-dimensional spin manifolds with odd $\hat{A}$-genus have nonvanishing $\alpha$-invariant and admit free $S^1$ actions. An example of such eight-dimensional spin manifolds can be found in [M]. (Note that the [generalized] Rochlin theorem implies that there are no $(8k+4)$-dimensional closed spin manifolds with odd $\hat{A}$-genus).

Received by the editors December 5, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58G10; Secondary 55N15.
*Key words and phrases.* $\alpha$-invariant, Dirac operator.
Partially supported by Grants-in-Aid for Encouragement of Young Scientists, The Ministry of Education, Science, and Culture, Japan.

The purpose of this note is to show the following:

**Theorem.** *Let $M$ be a closed spin manifold of $8k+1$ or $8k+2$ dimension. If $M$ admits an odd $S^1$ action, then the $\alpha$-invariant $\alpha(M)$ is zero.*

Ochanine and Landweber and Stong proved that the index of the Dirac operator twisted by certain vector bundles vanishes if the action is semifree and of odd type [L-S], [O]. To prove the above theorem, we show the vanishing of the index of the real Dirac operator.

## 1. Spin representations and the real Dirac operator

First of all, we review the real Dirac operator. The Clifford algebra $\mathrm{Cl}(U)$ associated with $(8k+1)$- or $(8k+2)$-dimensional Euclidean vector spaces $U$ is a simple algebra, and there is a unique irreducible graded representation of $\mathrm{Cl}(U)$. $V = V^0 \oplus V^1$ denotes the representation space. Associated with this representation and the spin structure $P$ we get a graded vector bundle $E = E^0 \oplus E^1$, which is a bundle of modules over the Clifford algebra bundle. The real Dirac operator is defined as follows:

$$D = \sum e_i \cdot \nabla_{e_i}$$

where $\{e_i\}$ is a local orthonormal frame of $M$ and $e_i \cdot$ is Clifford multiplication by $e_i$.

In case of $8k+1$ dimension, the $\alpha$-invariant equals $\dim_{\mathbf{R}} \ker(D\colon \Gamma(E^0) \to \Gamma(E^1)) \pmod 2$. In case of $8k+2$ dimension, $V = V^0 \oplus V^1$ has the complex structure induced from Clifford multiplication by $u_1 \cdots u_{8k+2}$, where $\{u_j\}$ is an oriented orthonormal basis of $U$. Thus $E = E^0 \oplus E^1$ has complex vector bundle structure. The $\alpha$-invariant equals $\dim_{\mathbf{C}} \ker(D\colon \Gamma(E^0) \to \Gamma(E^1)) \pmod 2$ [A-S], [A-S V].

## 2. Rigidity of the Dirac operator

Atiyah and Hirzebruch proved the vanishing of the equivariant index of the Dirac operator [A-H]. $M$ denotes a closed spin manifold with $S^1$ action. By taking the double covering action, the $S^1$ action may be assumed to be lifted to the spin structure $P$. The Dirac operator $D^{\pm}$ acting on $\pm$ spinor bundles $S^{\pm}$ is $S^1$-invariant; hence, we can define the equivariant index $\mathrm{ind}(D) \in R(S^1)$. The Atiyah and Hirzebruch theorem says $\mathrm{ind}(D) = 0$ in $R(S^1)$. If $M$ is $(8k+2)$-dimensional, the $\pm$ spinor bundle $S^{\pm}$ are nothing but the real spinor bundle $E = E^0 \oplus E^1$, with the complex vector bundle structure mentioned above. Hence the underlying real vector bundles of $S^+$ and $S^-$ are the same and the complex vector bundle structures are opposite to each other.

**Lemma.** *Let $M$ be an $(8k+2)$-dimensional closed spin manifold with $S^1$ action of even type. We write $\ker(D^+) = \sum a_k t^k$ and $\ker(D^-) = \sum b_k t^k$ in $R(S^1) \cong \mathbf{Z}[t, t^{-1}]$. Then we have*

$$a_k = a_{-k} \quad \text{and} \quad b_k = b_{-k}.$$

*Proof*. By the Atiyah-Hirzebruch theorem, we have $\sum a_k t^k - \sum b_k t^k = 0$; i.e., $a_k = b_k$ for any $k$. Since $S^+$ and $S^-$ are complex conjugates, $a_k = b_{-k}$ holds for any $k$. Hence we get the conclusion.

*Remarks.*

(1) We regard this statement as the rigidity property of the Dirac operator on $8k+2$-dimensional spin manifolds. Here the rigidity means that the equivariant index in $KO_{S^1}^{-2}(pt) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}[t] \cdot t$ belongs to the image of the homomorphism $KO^{-2}(pt) \to KO_{S^1}^{-2}(pt)$.

(2) Since the homomorphism $KO^{-1}(pt) \to KO_{S^1}^{-1}(pt)$ is an isomorphism, the rigidity property of the Dirac operator in the above sense holds automatically in dimension $8k+1$.

## 3. Proof of theorem

Let $M$ be a closed spin manifold with an odd $S^1$ action and $E = E^0 \oplus E^1$ the real spinor bundle (§1). By taking the double covering action, the $S^1$ action becomes of even type and the space of harmonic spinors is an $S^1$ module.

1. $(8k+1)$-dimensional case. The space of $\ker(D\colon \Gamma(E^0) \to \Gamma(E^1))$ decomposes into the direct sum of $S^1$ modules $\oplus W_l$, where $l$ is the weight of $W_l$ as a real $S^1$ module. Since $W_l$, $l \neq 0$, is even-dimensional, $\alpha(M) = \dim_{\mathbf{R}} W_0 \pmod 2$. $S^1$ acts trivially on $W_0$; especially, $-1$ acts trivially. On the other hand, we are considering the double covering action so that $-1$ acts as an identity on $M$. Hence it can be seen as the gauge transformation of $P$ and turns out $-1$ in $P \times_{Ad} \mathrm{Spin}(m) \subset \mathrm{Cl}(M)$. Hence $-1$ acts on $E$ as $(-1)$ times identity; especially, $-1$ acts on $W_0$ as the multiplication by $-1$. Therefore, we get $W_0 = 0$, which implies the conclusion.

2. $(8k+2)$-dimensional case. The space $\ker(D\colon \Gamma(E^0) \to \Gamma(E^1))$ decomposes into the direct sum of complex $S^1$ modules $\sum N_k$, where $k$ is the weight of $N_k$ as a complex $S^1$ module. By the lemma in §2, $N_k$ and $N_{-k}$ have the same dimension. Hence we get $\alpha(M) = \dim_{\mathbf{C}} N_0 \pmod 2$. The rest of the proof continues in a way similar to that of the case of $8k+1$ dimension.

*Remark.* The statement for $8k+1$ dimension follows from the one for $8k+2$ dimension. Let $M$ be an $(8k+1)$-dimensional spin manifold with an odd $S^1$ action. Consider the product spin manifold of $M$ and $S^1$ with product spin structure. $M \times S^1$ admits an odd $S^1$ action, and $\alpha(M \times S^1) = \alpha(M)$ holds.

## References

[A-B-S] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology **3** (1964), 3–38.

[A-H] M. F. Atiyah and F. Hirzebruch, *Spin manifolds and group actions*, Essays in Topology and Related Topics, Springer-Verlag, Berlin, 1970, pp. 18–28.

[A-S] M. F. Atiyah and I. M. Singer, *Index theory for skew adjoint Fredholm operators*, Publ. Math. IHES **37** (1969) 5–26.

[A-S V] ——, *The index of elliptic operators V*, Ann. Math. **93** (1971), 139–149.

[H] N. Hitchin, *Harmonic spinors*, Adv. in Math. **14** (1974), 1–55.

[L-S] P. S. Landweber and R. E. Stong, *Circle actions on spin manifolds and characteristic numbers*, Topology **27** (1988), 145–161.

[L] A. Lichnerowicz, *Spineurs harmoniques*, C.R. Acad. Sci. Paris Sér. A-B **257** (1963), 7–9.

[M] J. Milnor, *Remarks concerning spin manifolds*, Differential and Combinatorial Topology: A Symposium in Honor of Marston Morse, Princeton Univ. Press, Princeton, NJ, 1965, pp. 55–62.

[O] S. Ochanine, *Genres elliptiques equivariants*, Elliptic Curves and Modular Forms in Algebraic Topology (P. S. Landweber, ed.), Lecture Notes in Math., vol. 1326, Springer-Verlag, Berlin, 1988, pp. 107–122.

Mathematical Institute, Tohoku University, Sendai 980, Japan

*Current address*: Department of Mathematics, Faculty of Science, Ochanomizu University, Otsuka, Tokyo 112, Japan