A REMARK ON INCOMPARABLE ULTRAFILTERS IN THE RUDIN-KEISLER ORDER

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Abstract. If $2^{<\kappa} > \kappa$ and $p$ is an ultrafilter on $\omega$ of character $\kappa$ then there exist many ultrafilters that are incomparable with $p$ in the Rudin-Keisler order.

The Rudin-Keisler order $\leq_{\text{RK}}$ on $\omega^*$ is defined as $p \leq_{\text{RK}} q$ iff there exists a function $f: \omega \to \omega$ such that $f(A) = p$, hence $A \in p \iff f^{-1}[A] \in q$.

Problem 19 from [vM] and problem 48 from [HvM] asks: does there exist for every $p \in \omega^*$ a $q \in \omega^*$ such that $p$ and $q$ are incomparable in the Rudin-Keisler order? A partial answer is given by Hindman in [H]: if $p$ has character $\kappa$ and either $\kappa$ is singular or all predecessors of $p$ are of character $\kappa$ then such a $q$ exists.

We shall obtain a stronger result as a corollary to the following theorem.

Theorem. If $\kappa < \kappa$ and $p \in \omega^*$ has character $\kappa$ then there exist $2^\kappa$ ultrafilters which are not $\leq_{\text{RK}}$-greater than $p$.

Before we give the proof we fix some notation.

Let $F$ denote the Fréchet (cofinite) filter on $\omega$.

For $\mathcal{F} \subseteq \mathcal{P}(\omega)$ put

$$\langle \mathcal{F} \rangle = \left\{ A \subseteq \omega : \exists H \in [\mathcal{F}]^{<\omega} \cap H \subseteq A \right\}. $$

We call the system $\mathcal{F}$ centered if $|\bigcap H| = \omega$ for every finite $H \subseteq \mathcal{F}$.

Clearly, if $\mathcal{F}$ is centered then $\langle \mathcal{F} \rangle$ is the filter generated by $\mathcal{F}$.

Let $\kappa 2$ denote the product of $\kappa$ copies of 2, or equivalently the set of functions from $\kappa$ to 2. Of course $|\kappa 2| = 2^\kappa$.

Proof. We find $2^\kappa$ distinct ultrafilters that are not $\leq_{\text{RK}}$-greater than $p$.

Let $\{f_\gamma : \gamma < \kappa\}$ be an enumeration of the set of functions from $\omega$ to $\omega$.

We shall construct by induction on $\gamma < \kappa$ filters $\mathcal{F}_g^\gamma$ for $g \in \kappa 2$ such that

$$\chi(\mathcal{F}_g^\gamma) \leq \kappa + \gamma, $$

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(2) there is a set $G \in \mathcal{F}^{\gamma+1}_g$ such that $f_\gamma[G] \notin p$, and

(3) if $\gamma < \delta < c$ then $\mathcal{F}_g^{\gamma} \subseteq \mathcal{F}_g^{\delta}$.

We shall see that (1) ensures (2) can be satisfied. Note that (2) implies any ultrafilter $q_g$ extending $\bigcup_{\gamma < c} \mathcal{F}_g^{\gamma}$ will not be $\leq_{RK}$-above $p$.

Let $\{A_\beta : \beta < \kappa, i \in 2\}$ be an independent family of sets. This means that for every $\beta < \kappa$ we have $A_\beta^0 \cap A_\beta^0 = \emptyset$ and $A_\beta^0 \cup A_\beta^0 = \omega$ and that for every finite partial function $\varphi : \kappa \to 2$ the intersection $\bigcap_{\beta \in \text{dom} \varphi} A_\beta^{\varphi(\beta)}$ is infinite. Such families exist; see e.g., [vM, Lemma 3.3.2].

For every $g \in \kappa^2$ we consider the family $\mathcal{F}_g = \{A_\beta^{g(\beta)} : \beta < \kappa\}$. Each family $\mathcal{F}_g$ is centered since we only took sets from the independent family. We start with the distinct filters

$$\mathcal{F}_g^0 = \langle F \cup \mathcal{F}_g \rangle$$

for $g \in \kappa^2$.

Now suppose we are at stage $\gamma < c$ and take $g \in \kappa^2$. Here we use part of Hindman’s construction. Since $\chi(p) = c$ the system $\{f_\gamma[B] : B \in \mathcal{F}_g^{\gamma}\}$ does not generate $p$. Hence there is a set $E$ such that $\omega - E \in p$ and $\{f_\gamma[B] : B \in \mathcal{F}_g^{\gamma}\} \cup \{E\}$ is centered. It follows that $\mathcal{F}_g^{\gamma} \cup \{f_\gamma^{-1}[E]\}$ is also centered and we can add $G = f_\gamma^{-1}[E]$ to $\mathcal{F}_g^{\gamma}$, i.e., we let $F_g^{\gamma+1} = \langle \mathcal{F}_g^{\gamma} \cup \{f_\gamma^{-1}[E]\}\rangle$.

In case $\gamma$ is a limit we let $\mathcal{F}_g^{\gamma} = \bigcup_{\delta < \gamma} \mathcal{F}_g^{\delta}$.

If at any moment $\mathcal{F}_g^{\gamma}$ is an ultrafilter then we can stop. It cannot possibly be $\leq_{RK}$-above $p$ since its character is smaller than that of $p$.

Since clearly every ultrafilter has at most $c \leq_{RK}$-predecessors, the following is an immediate consequence of our theorem.

Corollary. If $\kappa < c$ is such that $2^\kappa > c$ then for every $p \in \omega^*$ of character $c$ there exist $2^c$ ultrafilters that are $\leq_{RK}$-incomparable with $p$.

Remarks. 1. Since $2^{cf} > c$ our corollary also covers the case when $c$ is singular.

2. Another easy consequence of our corollary is that if one assumes that $2^\kappa > c$ for some $\kappa < c$ and that there is a $p \in \omega^*$ such that $\chi(r) = c$ for all $r \leq_{RK} p$ then for every $u \in \omega^*$ there is a $v \in \omega^*$ that is $\leq_{RK}$-incomparable with it. (Given $u$, if the character of $u$ is $c$, the corollary applies. Otherwise $p$ is not $\leq_{RK}$-comparable to $u$.)

References


H N. Hindman, Is there a point of $\omega^*$ that sees all others?, Proc. Amer. Math. Soc. 104 (1988), 1235–1238.


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