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LOCAL SPECTRUM AND GENERALIZED SPECTRUM

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ABSTRACT. This paper provides the proofs of those results announced in [5, §5] that deal with the connection between the regular set and the local resolvant set of closed operators on a Hilbert space. We also give some characterizations and properties of Cowen–Douglas operators.

INTRODUCTION

Let H be a Hilbert space and let A be a closed operator with domain $D(A)$ and range $R(A)$ in H. $N(A)$ and $\sigma(A)$ will respectively denote the kernel and the spectrum of A.

If U is a subset of $\mathbb{C}$, we say that A has a generalized resolvent operator $Rg(A, \lambda)$ in U; if $\forall \lambda \in U$, $Rg(A, \lambda)$ is a bounded operator of H into $D(A)$ such that

$$(A - \lambda I)Rg(A, \lambda)(A - \lambda I) = (A - \lambda I)$$
$$Rg(A, \lambda)(A - \lambda I)Rg(A, \lambda) = Rg(A, \lambda)$$

[4–6] contain discussions of a generalization of the resolvent set $\rho(A)$ (the complement of $\sigma(A)$ with respect to $\mathbb{C}$), where the notion of inverse operator is replaced by that of generalized inverse operator. Let $\operatorname{reg}(A)$ denote the regular set of A defined by

$$\operatorname{reg}(A) = \{\lambda \in \mathbb{C};\ A \text{ has a generalized resolvent, analytic on a neighborhood } U \text{ of } \lambda\}.$$

It is clear that $\operatorname{reg}(A)$ is open and contains $\rho(A)$. The generalized spectrum of A, $\sigma_g(A)$ (the complement of $\operatorname{reg}(A)$ with respect to $\mathbb{C}$), has properties analogous to those of $\sigma(A)$ in classical spectral theory (cf. [5, 6]).

We shall say that A is regular if $R(A)$ is closed and $\forall n > 0$, $N(A^n) \subseteq R(A)$.

In [4, Theorem 2.6] it was shown that $\lambda \in \operatorname{reg}(A)$ if and only if $A - \lambda I$ is regular. The core of A, denoted by $\operatorname{Co}(A)$, is, by definition, the largest subspace M of H such that $A(M \cap D(A)) = M$. $\operatorname{Co}(A - \lambda I)$ is constant on each connected component of $\operatorname{reg}(A)$. If A is regular, then $\operatorname{Co}(A)$ is closed

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and $\mathrm{Co}(A) = \bigcap_{n\geq 0} R(A^n) = \bigcap_{n\geq 0} R(A-\lambda_n I)$, where $\{\lambda_n\}$ is a sequence of two by two distinct points of the component of $\mathrm{reg}(A)$ that contains zero and is convergent to zero (cf. [4]).

If $u \in H$, denote by $\delta_A(u)$ the set of $\mu \in \mathbb{C}$ such that there exists a neighborhood V_μ of μ in $\mathbb{C}$ and a function $f : V_\mu \to D(A)$ analytic on V_μ such that

$$\forall \lambda \in V_\mu, \qquad (A-\lambda I)f(\lambda) = u.$$

$\delta_A(u)$ is called the local resolvent set of A at u and its complement with respect to $\mathbb{C}$, denoted by $\gamma_A(u)$, is called the local spectrum of A at u (cf. [1, 7, 8]).

The first section of this paper provides the proofs of the results announced in [5, §5], that deal with the connection between the regular set and the local resolvent set of closed operators on a Hilbert space. It should be observed that Corollary 1.3 is a local version of [1, Chapter I, Proposition 3.7] and that Corollary 1.12 is a generalization of that same proposition.

In §2 of this paper, we give some characterizations and properties of Cowen–Douglas operators (Theorem 2.5 and Corollary 2.6).

1. Results

Theorem 1.1. *Let A be a closed operator; G a connected component of* $\mathrm{reg}(A)$; *and $\lambda_0 \in G$. Then*

$$G \subseteq \delta_A(u) \Leftrightarrow u \in \mathrm{Co}(A-\lambda_0 I).$$

Proof. $(\Rightarrow)\lambda_0 \in G \subseteq \delta_A(u)$. Then by definition, $\exists U_0$ a neighborhood of λ_0 in $\mathbb{C}$, and $f : U_0 \to D(A)$ is a function analytic on U_0 such that $\forall \lambda \in U_0$, $(A-\lambda I)f(\lambda) = u$. Hence $\forall \lambda \in U_0$, $u \in R(A-\lambda I)$, and, in particular, $u \in \bigcap_{j>0} R(A-\lambda_j I)$, where $\{\lambda_j\}$ is a sequence of points of $G \cap U_0$ two by two distinct that converges to λ_0. Therefore $u \in \mathrm{Co}(A-\lambda_0 I)$.

$(\Leftarrow)$ Let $\mu \in G$. By definition of $\mathrm{reg}(A)$, $\exists U_\mu \subseteq G$ a neighborhood of μ in $\mathbb{C}$ and a generalized resolvent operator $Rg(A, \lambda)$ of A analytic on U_μ. Furthermore $P_\lambda = (A-\lambda I)Rg(A, \lambda)$ is a projection such that $R(P_\lambda) = R(A-\lambda I)$. $\mathrm{Co}(A-\lambda I)$ is constant on G; hence, $\forall \lambda \in U_\mu$, $u \in \mathrm{Co}(A-\lambda_0 I) = \mathrm{Co}(A-\lambda I) \subseteq R(A-\lambda I)$, whence $\forall \lambda \in U_\mu$, $u = P_\lambda u$. Therefore $\forall \mu \in G$, $\exists U_\mu$ neighborhood of μ and $f(\lambda) = Rg(A, \lambda)u$ analytic on U_μ such that $\forall \lambda \in U_\mu$, $(A-\lambda I)f(\lambda) = (A-\lambda I)Rg(A, \lambda)u = P_\lambda u = u$, whence $\mu \in \delta_A(u)$, and therefore $G \subseteq \delta_A(u)$.

Corollary 1.2. *Let A be a closed operator, and let* $\mathrm{reg}(A) = \bigcup_{i\geq 0} G_i$. *Then, if $u \in H$,*

$$\mathrm{reg}(A) \subseteq \delta_A(u) \Leftrightarrow u \in \bigcap_{\lambda_i \in G_i} \mathrm{Co}(A-\lambda_i I).$$

Corollary 1.3. *Let A be a closed operator; G a connected component of* $\mathrm{reg}(A)$; *and $u \in H$. Then*

$$G \cap \delta_A(u) \neq \varnothing \Rightarrow G \subseteq \delta_A(u).$$

Proof. Let $\lambda_0 \in G \cap \delta_A(u)$. Then $u \in \mathrm{Co}(A - \lambda_0 I)$ (cf. [7, Proposition 1.3]). Using Theorem 1.1, it follows that $G \subseteq \delta_A(u)$.

Corollary 1.4. *Let A be a regular operator and G be the connected component of* $\mathrm{reg}(A)$ *that contains zero. Consider the equation*

$$\text{(E)} \qquad (A - \lambda I)x = y,$$

where $y \in H$ and $\lambda \in G$. Then

(1) *$y \in \mathrm{Co}(A)$ if and only if there exists a function $x(\lambda)$, having values in H and analytic in G, that is a solution of* (E).

(2) *If $y \notin \mathrm{Co}(A)$,* (E) *has a solution only for isolated λ in G.*

Proof. (1) is an immediate consequence of Theorem 1.1.

(2) Let $y \notin \mathrm{Co}(A)$; and suppose there exists a sequence $\{\lambda_n\}_{n\geq 1}$ of two by two distinct points of G that accumulate at a point $\lambda_0 \in G$, and that (E) has a solution for each λ_n. Then $y \in \bigcap_{n\geq 1} R(A - \lambda_n I) = \mathrm{Co}(A - \lambda_0 I)$, and since $\mathrm{Co}(A - \lambda I)$ is constant on G, it follows that $y \in \mathrm{Co}(A)$, a contradiction.

Remark 1.5. Set $H_1(A) = \bigcap_{\lambda_i \in G_i} \mathrm{Co}(A - \lambda_i I)$. Then the following inclusions are true:

$$\rho(A) \subseteq \bigcap_{u\in H} \delta_A(u) \subseteq \mathrm{reg}(A) \subseteq \bigcap_{u\in H_1(A)} \delta_A(u).$$

Indeed, the middle inclusion follows from the fact,

$$\bigcap_{u\in H} \delta_A(u) \subseteq \{\lambda \in \mathbb{C};\ R(A - \lambda I) = H\} \subseteq \mathrm{reg}(A).$$

The last inclusion follows from Corollary 1.2. It is a proper inclusion, as is shown by the following example:

Let $H = \ell^2$; $\{e_n\}$ be its natural basis; and A be the operator defined by $Ae_n = e_{n+1}$. It is easy to see that $N(A) = \{0\}$ and that $R(A)$ is closed. Hence, A is regular. Also $\mathrm{Co}(A) = \bigcap_{n\geq 0} R(A^n) = \{0\}$, hence $H_1(A) = 0\}$, and therefore $\bigcap_{u\in H_1(A)} \delta_A(u) = \mathbb{C}$. On the other hand, $\mathrm{reg}(A) = \mathbb{C}\backslash\Pi$ where Π is the unit circle.

Lemma 1.6. *Denote by $H_0(A) = \{u \in D^\infty(A); \lim_{n\to\infty} \|A^n u\|^{1/n} = 0\}$ the quasinilpotent part of A. Then*

(i) $\forall n \geq 0$, $N(A^n) \subseteq H_0(A)$.

(ii) *If A is regular then,* $\overline{H_0(A)} = \overline{\bigcup_{n\geq 0} N(A^n)} \subseteq \mathrm{Co}(A)$.

Proof. (i) Obvious. (ii) See [4, Proposition 2.10].

Proposition 1.7. *Let A be a densely defined closed operator on H such that A and A^* have the S.V.E.P. (single value extension property). Then*

$$\rho(A) = \mathrm{reg}(A).$$

Proof. It suffices to show that $\mathrm{reg}(A) \subseteq \rho(A)$. From [7, Corollary 1.5] it follows that if A has the S.V.E.P., then $\forall \lambda \in \mathrm{reg}(A)$, $N(A - \lambda I) = \{0\}$. By symmetry

between A and A^*, we see that $\forall\lambda \in \operatorname{reg}(A)$, $N(A^* - \bar{\lambda}I) = \{0\}$, and therefore $\operatorname{reg}(A) \subseteq \rho(A)$.

Corollary 1.8. *Let A be a densely defined closed operator on H such that A and A^* have the S.V.E.P. Then*

$$A \text{ is semi-Fredholm} \Rightarrow A \text{ is a Riesz–Schauder operator}.$$

Proof. If A is semi-Fredholm, then there exists a neighborhood U of the origin such that $\forall\lambda \in U\backslash\{0\}$, $A-\lambda I$ is regular (see [3, Proposition 4.3.1a]). According to the preceding proposition, it follows that $\forall\lambda \in U\backslash\{0\}$, $A - \lambda I$ is invertible. From the continuity of the index, we conclude that A is a Fredholm operator with index zero.

Moreover, if M and N provide a Kato decomposition of degree d of A, then $A_{|M}$ is regular and $A_{|N}$ is nilpotent of order d. Let us show that $M \cap H_0(A) = \operatorname{Co}(A) \cap H_0(A)$. Indeed $\operatorname{Co}(A) \subseteq M \Rightarrow \operatorname{Co}(A) \cap H_0(A) \subseteq M \cap H_0(A)$. Furthermore, $M \cap H_0(A) = H_0(A_{|M})$, and the fact that $A_{|M}$ is regular implies that $M \cap H_0(A) \subseteq \operatorname{Co}(A)$. Hence $M \cap H_0(A) = \operatorname{Co}(A) \cap H_0(A)$. But A has the S.V.E.P., which implies that $\operatorname{Co}(A) \cap H_0(A) = \{0\}$ (cf. [7, Corollary 1.5]), and hence that $M \cap H_0(A) = \{0\}$.

On the other hand, $N \subseteq N(A^d) \subseteq H_0(A) \Rightarrow H_0(A) = M \cap H_0(A) \oplus N$. Therefore $H_0(A) = N \subseteq N(A^d)$, whence $\forall n \geq d$, $N(A^n) \subseteq N(A^d)$ so that $N(A^d) = N(A^{d+1})$. Finally, by making use of the symmetry between A and A^*, we see that $R(A^d) = R(A^{d+1})$.

Proposition 1.9. *Let A be a regular operator and M a closed subspace of H invariant by A, such that $H_0(A) \subseteq M$. Then $A_{|M}$ is regular.*

Proof. Let us show first that $R(A_{|M}) = A(M \cap D(A))$ is closed. Observe that

$$N(A) \subseteq H_0(A) \subseteq M \Rightarrow M = N(A) \oplus M \cap N(A)^{\perp}. \tag{1}$$

Let $u_n \in R(A_{|M})$. Then $\exists v_n \in D(A) \cap M$ such that $u_n = Av_n$. It follows from (1) that the $v_n \perp N(A)$ can be chosen. Suppose that $u_n \to u$. Since $R(A)$ is closed, it follows that $v_n \to v$ and, since M is closed, $v \in M$, whence $u = Av \in R(A_{|M})$.

It remains to show that $\forall n \geq 0$, $N((A_{|M})^n) \subseteq R(A_{|M})$. Since A is regular, $\forall n \geq 0$, $N(A^n) \subseteq R(A)$, which implies that, if $u \in N(A^n)$, then $\exists v \in D(A)$ such that $u = Av \in H_0(A) \subseteq M$. Thus $u \in M$ and $v \in N(A^{n+1}) \subseteq M$, and therefore $\forall u \in N(A^n)$, we have $u \in R(A_{|M})$. Since $N((A_{|M})^n) \subseteq N(A^n)$, it follows that $\forall n \geq 0$, $N((A_{|M})^n) \subseteq R(A_{|M})$, and that $A_{|M}$ is regular.

Corollary 1.10. *Let A be a regular operator with $N(A) = \{0\}$. Then for every closed subspace M of H invariant by A, $A_{|M}$ is regular.*

Proof. Using Lemma 1.6(ii), we see that $R(A)$ is closed and that the equation $N(A) = \{0\}$ implies that $H_0(A) = \{0\}$. Hence, for every subspace M of H, $H_0(A) \subseteq M$, and the conclusion follows from the preceding proposition.

Corollary 1.11. *Let A be a closed operator that has the S.V.E.P. Then, for every closed subspace of H invariant by A,* $\operatorname{reg}(A) \subseteq \operatorname{reg}(A_{|M})$.

Proof. According to Proposition 1.7, if A has the S.V.E.P., then $\forall \lambda \in \operatorname{reg}(A)$, $N(A - \lambda I) = \{0\}$. The rest of the proof is a consequence of the previous corollary.

Corollary 1.12. *Let A be a closed operator that has the S.V.E.P., and let G be a connected component of* $\operatorname{reg}(A)$. *Then for every closed subspace of H invariant by A,*

$$G \cap \sigma(A_{|M}) \neq \varnothing \Rightarrow G \subseteq \sigma(A_{|M}).$$

Proof. According to Corollary 1.11, G is contained in a component of $\operatorname{reg}(A_{|M})$. Moreover, A has the S.V.E.P.. Then $\sigma(A_{|M}) = \bigcup_{u \in M} \gamma_{A_{|M}}(u)$. Now the proof follows from applying Corollary 1.3 to the operator $A_{|M}$.

Remark. This last result is a generalization of [1, Chapter I, Proposition 3.7].

2. Applications to Cowen–Douglas operators

The following lemmas will be needed:

Lemma 2.1. *Let A be a closed and regular operator with $D(A)$ dense on H. Then*

$$\operatorname{Co}(A^*) = H_0(A)^{\perp}.$$

Proof. A is regular and $D(A)$ is dense on H, implying that A^* is regular and $\forall j \geq 0$, $N(Aj)^{\perp} = R(A^{*j})$ (see [4, Proposition 2.8]). Let $A_1^* : \operatorname{Co}(A^*) \to \operatorname{Co}(A^*)$, the restriction of A^* to $\operatorname{Co}(A^*)$. Then A_1^* is onto, and there is a bound operator B of $\operatorname{Co}(A^*)$ that takes its values in $\operatorname{Co}(A^*) \cap D(A^*)$ such that $\forall u \in \operatorname{Co}(A^*)$, $A_1^* B u = u$. Consequently, $\forall u \in \operatorname{Co}(A^*)$, $\exists v_n = B^n u$ such that $u = A^{*n} v_n$ and $\|v_n\| \leq \|B\|^n \|u\|$ for all $n \geq 0$.

Let $u \in \operatorname{Co}(A^*)$ and $w \in H_0(A)$. Then $(w, u) = (w, A^{*n} v_n) = (A^n w, v_n)$. Therefore, $\forall n \geq 0$, $|(w, u)| \leq \|A^n w\| \|v_n\| \leq \|B\|^n \|A^n w\| \|u\|$. Since $w \in H_0(A)$, then $\lim_{n \to \infty} \|B\|^n \|A^n w\| = 0$. Therefore, $(w, u) = 0$. This shows that $H_0(A) \subseteq \operatorname{Co}(A^*)^{\perp}$ or $\operatorname{Co}(A^*) \subseteq H_0(A)^{\perp}$.

Inversely, $\forall j \geq 0$, $N(A^j) \subseteq H_0(A)$ whence $\forall j \geq 0$, $H_0(A)^{\perp} \subseteq N(A^j)^{\perp} = R(A^{*j})$. Therefore $H_0(A)^{\perp} \subseteq \bigcap_{j \geq 0} R(A^{*j}) = \operatorname{Co}(A^*)$.

Lemma 2.2. *Let A be a closed and regular operator with $D(A)$ dense on H. Then the map $\lambda \mapsto \overline{H_0(A - \lambda I)}$ is constant on each connected component of* $\operatorname{reg}(A)$.

Proof. The proof of this lemma is a consequence of Lemma 2.1 and the fact that the core is constant within every connected component of $\operatorname{reg}(A)$.

Lemma 2.3. *Let A be a closed and regular operator with $D(A)$ dense on H and Ω a connected component of* $\operatorname{reg}(A)$ *that contains zero. Then*

$$\overline{H_0(A)} = \text{c.l.m.}\{N(A - \lambda I)\}_{\lambda \in \Omega} = \text{c.l.m.}\{N(A^n)\}_{n \geq 0},$$

where c.l.m. *denotes closed linear manifold.*

Proof. Observe that $\forall\lambda \in \mathbb{C}$, $N(A-\lambda I) \subseteq H_0(A-\lambda I)$. Using Lemma 2.2, we deduce that $\forall\lambda \in \Omega$, $N(A-\lambda I) \subseteq \overline{H_0(A)}$. Consequently c.l.m. $\{N(A-\lambda I)\}_{\lambda\in\Omega} \subseteq \overline{H_0(A)}$.

Inversely, let $u \in [\text{c.l.m.}\{N(A-\lambda I)\}_{\lambda\in\Omega}]^{\perp}$, particularly $u \in N(A-\lambda_i I)^{\perp}$ where $\{\lambda_i\}_{i>0}$ is a sequence of two by two distinct points of Ω and is convergent to zero. Therefore $u \in R(A^* - \bar{\lambda}_i I)$, which implies $u \in \bigcap_{i>0} R(A^* - \bar{\lambda}_i I) = \mathrm{Co}(A^*)$. Hence $[\text{c.l.m.}\{N(A-\lambda I)\}_{\lambda\in\Omega}]^{\perp} \subseteq \mathrm{Co}(A^*) = H_0(A)^{\perp}$, and therefore $\overline{H_0(A)} \subseteq \text{c.l.m.}\{N(A-\lambda I)\}_{\lambda\in\Omega}$.

See [4, Proposition 2.10] for the proof of the second equality in this lemma.

Definition 2.4 (see [2]). We say that $T \in B(H)$ is a Cowen–Douglas operator if there exist a connected open set Ω of $\mathbb{C}$ and a positive integer n such that

(1) $\Omega \subseteq \sigma(T)$.
(2) $R(T-\omega I) = H$, $\forall\omega \in \Omega$.
(3) $\dim N(T-\omega I) = n$, $\forall\omega \in \Omega$.
(4) $\text{c.l.m.}\{N(T-\omega I)\}_{\omega\in\Omega} = H$.

In this case, we shall denote $T \in B_n(\Omega)$.

Remark. (2) and (4) $\Rightarrow$ (1); indeed, (2) $\Rightarrow$ $\Omega \subseteq \mathrm{reg}(T)$ and (4) $\Rightarrow$ $H_0(T-\omega I) = H$. Therefore $N(T-\omega I) \neq \{0\}$ (see Lemma 2.3); consequently, $\Omega \subseteq \sigma(T)$.

Theorem 2.5. *Let $T \in B(H)$. Then the following conditions are equivalent*:

(i) *there exists a connected open set Ω of $\mathbb{C}$ containing zero and a positive integer n, such that $T \in B_n(\Omega)$.*
(ii) $\mathrm{Co}(T) = H$, $\overline{H_0(T)} = H$, *and* $\dim N(T) = n$.
(iii) $\mathrm{Co}(T) = H$, $\mathrm{Co}(T^*) = \{0\}$, *and* $\dim N(T) = n$.
(iv) T^* *regular,* $\mathrm{Co}(T^*) = \{0\}$, *and* $\mathrm{codim}\, R(T^*) = n$.

Proof. (i) $\Rightarrow$ (ii) is an immediate consequence of Lemma 2.3 and Definition 2.4.

(ii) $\Rightarrow$ (iii) is a consequence of Lemma 2.1.

(iii) $\Rightarrow$ (iv) is obvious.

For (iv) $\Rightarrow$ (i), note that T^* regular $\Rightarrow N(T^*) \subseteq \mathrm{Co}(T^*) = \{0\}$ and $R(T^*)$ closed. It is easy to see that $R(T) = H$ and that $\mathrm{Co}(T) = H$. Since $\mathrm{Co}(T) = H$, T is regular and there exists a neighborhood Ω of the origin such that $\forall\omega \in \Omega$, $T-\omega I$ is regular, and $R(T-\omega I) = H$. The rest of the proof is a consequence of Lemma 2.1, Lemma 2.3, and [6, Theorem 2.1].

Corollary 2.6. *Let $T \in B_n(\Omega)$. Then*

(1) $\sigma(T)$ *is connected,*
(2) $\sigma_w(T) = \sigma(T)$,
(3) $\sigma_p(T^*) = \varnothing$,
(4) $\mathrm{ind}(T-\lambda I) \geq 0$, $\forall\lambda \in \rho_{S-F}(T)$,
(5) T^* *has the S.V.E.P. and T does not have the S.V.E.P. at all $\omega \in \Omega$,*

where $\rho_{S-F}(T) = \{\lambda \in \mathbb{C};\ T - \lambda I$ *is semi-Fredholm*$\}$ *and* $\sigma_w(T) = \{\lambda \in \mathbb{C};\ T - \lambda I$ *is not semi-Fredholm of index 0*$\}$, *is the Weyl spectrum of* T.

Proof. Without loss of generality, we can assume that $0 \in \Omega$. (1), (2), and (3) are deduced from [6, §2] and Theorem 2.5, since $\mathrm{Co}(T^*) = \{0\}$ and $N(T^*) = \{0\}$.

(4) is a consequence of (3).

(5) $\sigma_p(T^*) = \varnothing \Rightarrow T^*$ has the S.V.E.P.. Moreover, if T has the S.V.E.P. in Ω, then Proposition 1.7 implies $\Omega \subseteq \rho(T)$.

Corollary 2.7. *Let* Ω *be a connected open set of* $\mathbb{C}$ *and* n *a positive integer. Then* $B_n(\Omega) \cap \{T \in B(H);\ T$ *is decomposable*$\} = \varnothing$.

Proof. From Theorem 2.5, $T \in B_n(\Omega) \Rightarrow \mathrm{Co}(T^*) = \{0\}$. On the other hand, the fact that T is decomposable implies that T^* is decomposable. Using [7, Corollary 2.2(ii)], we deduce that T^* is quasinilpotent; therefore, $\sigma(T) = \{0\}$, which is a contradiction, because $\Omega \subseteq \sigma(T)$.

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