LOCAL SPECTRUM AND GENERALIZED SPECTRUM

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Abstract. This paper provides the proofs of those results announced in [5, §5] that deal with the connection between the regular set and the local resolvent set of closed operators on a Hilbert space. We also give some characterizations and properties of Cowen–Douglas operators.

Introduction

Let $H$ be a Hilbert space and let $A$ be a closed operator with domain $D(A)$ and range $R(A)$ in $H$. $N(A)$ and $\sigma(A)$ will respectively denote the kernel and the spectrum of $A$.

If $U$ is a subset of $\mathbb{C}$, we say that $A$ has a generalized resolvent operator $Rg(A, \lambda)$ in $U$; if $\forall \lambda \in U$, $Rg(A, \lambda)$ is a bounded operator of $H$ into $D(A)$ such that

$$(A - \lambda I)Rg(A, \lambda)(A - \lambda I) = (A - \lambda I)$$

$Rg(A, \lambda)(A - \lambda I)Rg(A, \lambda) = Rg(A, \lambda)

[4–6] contain discussions of a generalization of the resolvent set $\rho(A)$ (the complement of $\sigma(A)$ with respect to $\mathbb{C}$), where the notion of inverse operator is replaced by that of generalized inverse operator. Let $\text{reg}(A)$ denote the regular set of $A$ defined by

$$\text{reg}(A) = \{ \lambda \in \mathbb{C} ; \text{A has a generalized resolvent, analytic on a neighborhood U of } \lambda \} .$$

It is clear that $\text{reg}(A)$ is open and contains $\rho(A)$. The generalized spectrum of $A$, $\sigma_{g}(A)$ (the complement of $\text{reg}(A)$ with respect to $\mathbb{C}$), has properties analogous to those of $\sigma(A)$ in classical spectral theory (cf. [5, 6]).

We shall say that $A$ is regular if $R(A)$ is closed and $\forall n > 0$, $N(A^n) \subseteq R(A)$.

In [4, Theorem 2.6] it was shown that $\lambda \in \text{reg}(A)$ if and only if $A - \lambda I$ is regular. The core of $A$, denoted by $\text{Co}(A)$, is, by definition, the largest subspace $M$ of $H$ such that $A(M \cap D(A)) = M$. $\text{Co}(A - \lambda I)$ is constant on each connected component of $\text{reg}(A)$. If $A$ is regular, then $\text{Co}(A)$ is closed.
and \( \text{Co}(A) = \bigcap_{n\geq0} R(A^n) = \bigcap_{n\geq0} R(A - \lambda_n I) \), where \( \{\lambda_n\} \) is a sequence of two by two distinct points of the component of \( \text{reg}(A) \) that contains zero and is convergent to zero (cf. [4]).

If \( u \in H \), denote by \( \delta_A(u) \) the set of \( \mu \in \mathbb{C} \) such that there exists a neighborhood \( V_\mu \) of \( \mu \) in \( \mathbb{C} \) and a function \( f : V_\mu \to D(A) \) analytic on \( V_\mu \) such that
\[
\forall \lambda \in V_\mu, \quad (A - \lambda I)f(\lambda) = u.
\]
\( \delta_A(u) \) is called the local resolvent set of \( A \) at \( u \) and its complement with respect to \( \mathbb{C} \), denoted by \( \gamma_A(u) \), is called the local spectrum of \( A \) at \( u \) (cf. [1, 7, 8]).

The first section of this paper provides the proofs of the results announced in [5, §5], that deal with the connection between the regular set and the local resolvent set of closed operators on a Hilbert space. It should be observed that Corollary 1.3 is a local version of [1, Chapter I, Proposition 3.7] and that Corollary 1.12 is a generalization of that same proposition.

In §2 of this paper, we give some characterizations and properties of Cowen–Douglas operators (Theorem 2.5 and Corollary 2.6).

1. Results

**Theorem 1.1.** Let \( A \) be a closed operator; \( G \) a connected component of \( \text{reg}(A) \); and \( \lambda_0 \in G \). Then
\[
G \subseteq \delta_A(u) \iff u \in \text{Co}(A - \lambda_0 I).
\]

**Proof.** (\( \Rightarrow \)) \( \lambda_0 \in G \subseteq \delta_A(u) \). Then by definition, \( \exists U_0 \) a neighborhood of \( \lambda_0 \) in \( \mathbb{C} \), and \( f : U_0 \to D(A) \) is a function analytic on \( U_0 \) such that \( \forall \lambda \in U_0, \quad (A - \lambda I)f(\lambda) = u \). Hence \( \forall \lambda \in U_0, \quad u \in R(A - \lambda I) \), and, in particular, \( u \in \bigcap_{j>0} R(A - \lambda_j I) \), where \( \{\lambda_j\} \) is a sequence of points of \( G \cap U_0 \) two by two distinct that converges to \( \lambda_0 \). Therefore \( u \in \text{Co}(A - \lambda_0 I) \).

(\( \Leftarrow \)) Let \( \mu \in G \). By definition of \( \text{reg}(A) \), \( \exists U_\mu \subseteq G \) a neighborhood of \( \mu \) in \( \mathbb{C} \) and a generalized resolvent operator \( Rg(A, \lambda) \) of \( A \) analytic on \( U_\mu \). Furthermore \( P_\lambda = (A - \lambda I)Rg(A, \lambda) \) is a projection such that \( R(P_\lambda) = R(A - \lambda I) \). \( \text{Co}(A - \lambda I) \) is constant on \( G \); hence, \( \forall \lambda \in U_\mu, \quad u \in \text{Co}(A - \lambda_0 I) = \text{Co}(A - \lambda I) \subseteq R(A - \lambda I), \) whence \( \forall \lambda \in U_\mu, \quad u = P_\lambda u \). Therefore \( \forall \mu \in G \), \( \exists U_\mu \) neighborhood of \( \mu \) and \( f(\lambda) = Rg(A, \lambda)u \) analytic on \( U_\mu \) such that \( \forall \lambda \in U_\mu \), \( (A - \lambda I)f(\lambda) = (A - \lambda I)Rg(A, \lambda)u = P_\lambda u = u \), whence \( \mu \in \delta_A(u) \), and therefore \( G \subseteq \delta_A(u) \).

**Corollary 1.2.** Let \( A \) be a closed operator, and let \( \text{reg}(A) = \bigcup_{l\geq0} G_l \). Then, if \( u \in H \),
\[
\text{reg}(A) \subseteq \delta_A(u) \iff u \in \bigcap_{\lambda_j \in G_l} \text{Co}(A - \lambda_j I).
\]

**Corollary 1.3.** Let \( A \) be a closed operator; \( G \) a connected component of \( \text{reg}(A) \); and \( u \in H \). Then
\[
G \cap \delta_A(u) \neq \emptyset \Rightarrow G \subseteq \delta_A(u).
\]
Proof. Let $\lambda_0 \in G \cap \delta_A(u)$. Then $u \in \text{Co}(A - \lambda_0 I)$ (cf. [7, Proposition 1.3]). Using Theorem 1.1, it follows that $G \subseteq \delta_A(u)$.

Corollary 1.4. Let $A$ be a regular operator and $G$ be the connected component of $\text{reg}(A)$ that contains zero. Consider the equation

$$\text{E: } (A - \lambda I)x = y,$$

where $y \in H$ and $\lambda \in G$. Then

1. $y \in \text{Co}(A)$ if and only if there exists a function $x(\lambda)$, having values in $H$ and analytic in $G$, that is a solution of (E).
2. If $y \notin \text{Co}(A)$, (E) has a solution only for isolated $\lambda$ in $G$.

Proof. (1) is an immediate consequence of Theorem 1.1.

(2) Let $y \notin \text{Co}(A)$; and suppose there exists a sequence $\{\lambda_n\}_{n \geq 1}$ of two by two distinct points of $G$ that accumulate at a point $\lambda_0 \in G$, and that (E) has a solution for each $\lambda_n$. Then $y \in \bigcap_{n \geq 1} R(A - \lambda_n I) = \text{Co}(A - \lambda_0 I)$, and since $\text{Co}(A - \lambda I)$ is constant on $G$, it follows that $y \notin \text{Co}(A)$, a contradiction.

Remark 1.5. Set $H_1(A) = \bigcap_{\lambda_1 \in G} \text{Co}(A - \lambda_1 I)$. Then the following inclusions are true:

$$\rho(A) \subseteq \bigcap_{u \in H} \delta_A(u) \subseteq \text{reg}(A) \subseteq \bigcap_{u \in H_1(A)} \delta_A(u).$$

Indeed, the middle inclusion follows from the fact,

$$\bigcap_{u \in H} \delta_A(u) \subseteq \{\lambda \in \mathbb{C}; R(A - \lambda I) = H\} \subseteq \text{reg}(A).$$

The last inclusion follows from Corollary 1.2. It is a proper inclusion, as is shown by the following example:

Let $H = l^2; \{e_n\}$ be its natural basis; and $A$ be the operator defined by $Ae_n = e_{n+1}$. It is easy to see that $N(A) = \{0\}$ and that $R(A)$ is closed. Hence, $A$ is regular. Also $\text{Co}(A) = \bigcap_{n \geq 1} R(A^n) = \{0\}$, hence $H_1(A) = 0$, and therefore $\bigcap_{u \in H(A)} \delta_A(u) = \mathbb{C}$. On the other hand, $\text{reg}(A) = \mathbb{C} \setminus \Pi$ where $\Pi$ is the unit circle.

Lemma 1.6. Denote by $H_0(A) = \{u \in D^\infty(A); \lim_{n \to \infty} \|A^n u\|^{1/n} = 0\}$ the quasinilpotent part of $A$. Then

(i) $\forall n \geq 0, N(A^n) \subseteq H_0(A)$.

(ii) If $A$ is regular then, $H_0(A) = \bigcup_{n \geq 0} N(A^n) \subseteq \text{Co}(A)$.

Proof. (i) Obvious. (ii) See [4, Proposition 2.10].

Proposition 1.7. Let $A$ be a densely defined closed operator on $H$ such that $A$ and $A^*$ have the S.V.E.P. (single value extension property). Then

$$\rho(A) = \text{reg}(A).$$

Proof. It suffices to show that $\text{reg}(A) \subseteq \rho(A)$. From [7, Corollary 1.5] it follows that if $A$ has the S.V.E.P., then $\forall \lambda \in \text{reg}(A), N(A - \lambda I) = \{0\}$. By symmetry
between $A$ and $A^*$, we see that $\forall \lambda \in \text{reg}(A), \ N(A^* - \lambda I) = \{0\}$, and therefore $\text{reg}(A) \subseteq \rho(A)$.

**Corollary 1.8.** Let $A$ be a densely defined closed operator on $H$ such that $A$ and $A^*$ have the S.V.E.P. Then

$A$ is semi-Fredholm $\Rightarrow$ $A$ is a Riesz–Schauder operator.

**Proof.** If $A$ is semi-Fredholm, then there exists a neighborhood $U$ of the origin such that $\forall \lambda \in U \setminus \{0\}, \ A - \lambda I$ is regular (see [3, Proposition 4.3.1a]). According to the preceding proposition, it follows that $\forall \lambda \in U \setminus \{0\}, \ A - \lambda I$ is invertible. From the continuity of the index, we conclude that $A$ is a Fredholm operator with index zero.

Moreover, if $M$ and $N$ provide a Kato decomposition of degree $d$ of $A$, then $A_{|M}$ is regular and $A_{|N}$ is nilpotent of order $d$. Let us show that $M \cap H_0(A) = \text{Co}(A) \cap H_0(A)$. Indeed $\text{Co}(A) \subseteq M \Rightarrow \text{Co}(A) \cap H_0(A) \subseteq M \cap H_0(A)$. Furthermore, $M \cap H_0(A) = H_0(A_{|M})$, and the fact that $A_{|M}$ is regular implies that $M \cap H_0(A) \subseteq \text{Co}(A)$. Hence $M \cap H_0(A) = \text{Co}(A) \cap H_0(A)$. But $A$ has the S.V.E.P., which implies that $\text{Co}(A) \cap H_0(A) = \{0\}$ (cf. [7, Corollary 1.5]), and hence that $M \cap H_0(A) = \{0\}$.

On the other hand, $N \subseteq N(A^n) \subseteq H_0(A) \Rightarrow H_0(A) = M \cap H_0(A) \oplus N$. Therefore $H_0(A) = N \subseteq N(A^n)$, whence $\forall n \geq d, \ N(A^n) \subseteq N(A^d)$ so that $N(A^n) = N(A^{d+1})$. Finally, by making use of the symmetry between $A$ and $A^*$, we see that $R(A^d) = R(A^{d+1})$.

**Proposition 1.9.** Let $A$ be a regular operator and $M$ a closed subspace of $H$ invariant by $A$, such that $H_0(A) \subseteq M$. Then $A_{|M}$ is regular.

**Proof.** Let us show first that $R(A_{|M}) = A(M \cap D(A))$ is closed. Observe that

\[(1) \quad N(A) \subseteq H_0(A) \subseteq M \Rightarrow M = N(A) \oplus M \cap N(A) .\]

Let $u_n \in R(A_{|M})$. Then $\exists v_n \in D(A) \cap M$ such that $u_n = Av_n$. It follows from (1) that the $v_n \perp N(A)$ can be chosen. Suppose that $u_n \rightarrow u$. Since $R(A)$ is closed, it follows that $v_n \rightarrow v$ and, since $M$ is closed, $v \in M$, whence $u = Av \in R(A_{|M})$.

It remains to show that $\forall n \geq 0, \ N((A_{|M})^n) \subseteq R(A_{|M})$. Since $A$ is regular, $\forall n \geq 0, \ N(A^n) \subseteq R(A)$, which implies that, if $u \in N(A^n)$, then $\exists v \in D(A)$ such that $u = Av \in H_0(A) \subseteq M$. Thus $u \in M$ and $v \in N(A^{n+1}) \subseteq M$, and therefore $\forall u \in N(A^n)$, we have $u \in R(A_{|M})$. Since $N((A_{|M})^n) \subseteq N(A^n)$, it follows that $\forall n \geq 0, \ N((A_{|M})^n) \subseteq R(A_{|M})$, and that $A_{|M}$ is regular.

**Corollary 1.10.** Let $A$ be a regular operator with $N(A) = \{0\}$. Then for every closed subspace $M$ of $H$ invariant by $A$, $A_{|M}$ is regular.

**Proof.** Using Lemma 1.6(ii), we see that $R(A)$ is closed and that the equation $N(A) = \{0\}$ implies that $H_0(A) = \{0\}$. Hence, for every subspace $M$ of $H$, $H_0(A) \subseteq M$, and the conclusion follows from the preceding proposition.
Corollary 1.11. Let $A$ be a closed operator that has the S.V.E.P. Then, for every closed subspace of $H$ invariant by $A$, $\text{reg}(A) \subseteq \text{reg}(A_{|M})$.

Proof. According to Proposition 1.7, if $A$ has the S.V.E.P., then $\forall \lambda \in \text{reg}(A), N(A - \lambda I) = \{0\}$. The rest of the proof is a consequence of the previous corollary.

Corollary 1.12. Let $A$ be a closed operator that has the S.V.E.P., and let $G$ be a connected component of $\text{reg}(A)$. Then for every closed subspace of $H$ invariant by $A$,

$$G \cap \sigma(A_{|M}) \neq \emptyset \Rightarrow G \subseteq \sigma(A_{|M}).$$

Proof. According to Corollary 1.11, $G$ is contained in a component of $\text{reg}(A_{|M})$. Moreover, $A$ has the S.V.E.P.. Then $\sigma(A_{|M}) = \bigcup_{\mu \in M} \gamma_{A_{|M}}(\mu)$. Now the proof follows from applying Corollary 1.3 to the operator $A_{|M}$.

Remark. This last result is a generalization of [1, Chapter I, Proposition 3.7].

2. Applications to Cowen–Douglas operators

The following lemmas will be needed:

Lemma 2.1. Let $A$ be a closed and regular operator with $D(A)$ dense on $H$. Then

$$\text{Co}(A^*) = H_0(A)^\perp.$$  

Proof. $A$ is regular and $D(A)$ is dense on $H$, implying that $A^*$ is regular and $\forall j \geq 0$, $N(Aj)^\perp = R(A^j)$ (see [4, Proposition 2.8]). Let $A_1^* : \text{Co}(A^*) \to \text{Co}(A^*)$, the restriction of $A^*$ to $\text{Co}(A^*)$. Then $A_1^*$ is onto, and there is a bound operator $B$ of $\text{Co}(A^*)$ that takes its values in $\text{Co}(A^*) \cap D(A^*)$ such that $\forall u \in \text{Co}(A^*), A_1^*Bu = u$. Consequently, $\forall u \in \text{Co}(A^*), \exists v_n = B^n u$ such that $u = A^{jn}v_n$ and $\|v_n\| \leq \|B\|^n\|u\|$ for all $n \geq 0$.

Let $u \in \text{Co}(A^*)$ and $w \in H_0(A)$. Then $(w, u) = (w, A^nw, v_n)$. Therefore, $\forall n \geq 0$, $\|(w, u)\| \leq \|A^nw\| \|v_n\| \leq \|B\|^n\|A^nw\| \|u\|$. Since $w \in H_0(A)$, then $\lim_{n \to \infty} \|B\|^n\|A^nw\| = 0$. Therefore, $(w, u) = 0$. This shows that $H_0(A) \subseteq \text{Co}(A^*)^\perp$ or $\text{Co}(A^*) \subseteq H_0(A)^\perp$.

Inversely, $\forall j \geq 0$, $N(A^j) \subseteq H_0(A)$ whence $\forall j \geq 0$, $H_0(A)^\perp \subseteq N(A^j)^\perp = R(A^j)$. Therefore $H_0(A)^\perp \subseteq \cap_{j \geq 0} R(A^j) = \text{Co}(A^*)$.

Lemma 2.2. Let $A$ be a closed and regular operator with $D(A)$ dense on $H$. Then the map $\lambda \mapsto H_0(A - \lambda I)$ is constant on each connected component of $\text{reg}(A)$.

Proof. The proof of this lemma is a consequence of Lemma 2.1 and the fact that the core is constant within every connected component of $\text{reg}(A)$.

Lemma 2.3. Let $A$ be a closed and regular operator with $D(A)$ dense on $H$ and $\Omega$ a connected component of $\text{reg}(A)$ that contains zero. Then

$$H_0(A) = \text{c.l.m.}\{N(A - \lambda I)\}_{\lambda \in \Omega} = \text{c.l.m.}\{N(A^j)\}_{n \geq 0},$$

where c.l.m. denotes closed linear manifold.
Proof. Observe that $\forall \lambda \in \mathbb{C}, \ N(A - \lambda I) \subseteq H_0(A - \lambda I)$. Using Lemma 2.2, we deduce that $\forall \lambda \in \Omega, \ N(A - \lambda I) \subseteq H_0(A)$. Consequently c.l.m. $\{N(A - \lambda I)\}_{\lambda \in \Omega} \subseteq H_0(A)$.

Inversely, let $u \in \{\text{c.l.m.}\{N(A - \lambda I)\}_{\lambda \in \Omega}\}^{\perp}$, particularly $u \in N(A - \lambda I)^{\perp}$, where $\{\lambda_i\}_{i \geq 0}$ is a sequence of two by two distinct points of $\Omega$ and is convergent to zero. Therefore $u \in R(A^* - \lambda_i I)$, which implies $u \in \bigcap_{i \geq 0} R(A^* - \lambda_i I) = \text{Co}(A^*)$. Hence $\{\text{c.l.m.}\{N(A - \lambda I)\}_{\lambda \in \Omega}\}^{\perp} \subseteq \text{Co}(A^*) = H_0(A)^{\perp}$, and therefore $H_0(A) \subseteq \text{c.l.m.}\{N(A - \lambda I)\}_{\lambda \in \Omega}$.

See [4, Proposition 2.10] for the proof of the second equality in this lemma.

Definition 2.4 (see [2]). We say that $T \in B(H)$ is a Cowen–Douglas operator if there exist a connected open set $\Omega$ of $\mathbb{C}$ and a positive integer $n$ such that

1. $\Omega \subseteq \sigma(T)$.
2. $R(T - \omega I) = H$, $\forall \omega \in \Omega$.
3. $\dim N(T - \omega I) = n$, $\forall \omega \in \Omega$.
4. c.l.m. $\{N(T - \omega I)\}_{\omega \in \Omega} = H$.

In this case, we shall denote $T \in B_n(\Omega)$.

Remark. (2) and (4) $\Rightarrow$ (1); indeed, (2) $\Rightarrow$ $\Omega \subseteq \text{reg}(T)$ and (4) $\Rightarrow$ $H_0(T - \omega I) = H$. Therefore $N(T - \omega I) \neq \{0\}$ (see Lemma 2.3); consequently, $\Omega \subseteq \sigma(T)$.

Theorem 2.5. Let $T \in B(H)$. Then the following conditions are equivalent:

(i) there exists a connected open set $\Omega$ of $\mathbb{C}$ containing zero and a positive integer $n$, such that $T \in B_n(\Omega)$.

(ii) $\text{Co}(T) = H$, $H_0(T) = H$, and $\dim N(T) = n$.

(iii) $\text{Co}(T) = H$, $\text{Co}(T^*) = \{0\}$, and $\dim N(T) = n$.

(iv) $T^*$ regular, $\text{Co}(T^*) = \{0\}$, and $\dim N(T) = n$.

Proof. (i) $\Rightarrow$ (ii) is an immediate consequence of Lemma 2.3 and Definition 2.4.

(ii) $\Rightarrow$ (iii) is a consequence of Lemma 2.1.

(iii) $\Rightarrow$ (iv) is obvious.

For (iv) $\Rightarrow$ (i), note that $T^*$ regular $\Rightarrow N(T^*) \subseteq \text{Co}(T^*) = \{0\}$ and $R(T^*)$ closed. It is easy to see that $R(T) = H$ and that $\text{Co}(T) = H$. Since $\text{Co}(T) = H$, $T$ is regular and there exists a neighborhood $\Omega$ of the origin such that $\forall \omega \in \Omega$, $T - \omega I$ is regular, and $R(T - \omega I) = H$. The rest of the proof is a consequence of Lemma 2.1, Lemma 2.3, and [6, Theorem 2.1].

Corollary 2.6. Let $T \in B_n(\Omega)$. Then

1. $\sigma(T)$ is connected,
2. $\sigma_w(T) = \sigma(T)$,
3. $\sigma_p(T^*) = \emptyset$,
4. $\text{ind}(T - \lambda I) \geq 0$, $\forall \lambda \in \rho_{S-F}(T)$,
5. $T^*$ has the S.V.E.P. and $T$ does not have the S.V.E.P. at all $\omega \in \Omega$.
where $\rho_{S-F}(T) = \{ \lambda \in \mathbb{C}; T - \lambda I \text{ is semi-Fredholm} \}$ and $\sigma_w(T) = \{ \lambda \in \mathbb{C}; T - \lambda I \text{ is not semi-Fredholm of index 0} \}$, is the Weyl spectrum of $T$.

Proof. Without loss of generality, we can assume that $0 \in \Omega$. (1), (2), and (3) are deduced from [6, §2] and Theorem 2.5, since $\text{Co}(T^*) = \{0\}$ and $N(T^*) = \{0\}$

(4) is a consequence of (3).

(5) $\sigma_p(T^*) = \emptyset \Rightarrow T^*$ has the S.V.E.P. Moreover, if $T$ has the S.V.E.P. in $\Omega$, then Proposition 1.7 implies $\Omega \subseteq \rho(T)$.

Corollary 2.7. Let $\Omega$ be a connected open set of $\mathbb{C}$ and $n$ a positive integer. Then $B_n(\Omega) \cap \{ T \in B(H); T \text{ is decomposable} \} = \emptyset$.

Proof. From Theorem 2.5, $T \in B_n(\Omega) \Rightarrow \text{Co}(T^*) = \{0\}$. On the other hand, the fact that $T$ is decomposable implies that $T^*$ is decomposable. Using [7, Corollary 2.2(ii)], we deduce that $T^*$ is quasinilpotent; therefore, $\sigma(T) = \{0\}$, which is a contradiction, because $\Omega \subseteq \sigma(T)$.

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