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NOTE ON A THEOREM OF AVAKUMOVIĆ

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ABSTRACT. A short proof is given of a result due to Avakumović. More specifically the asymptotic behavior of the solution $y(x) \to 0$ $(x \to \infty)$ of the differential equation $y'' = \phi(x)y^\lambda$ $(\lambda > 1)$ in case $\phi(tx)/\phi(x) \to t^\sigma$ $(x \to \infty)$, $\sigma > -2$ is given.

In a paper published in 1947, Avakumović [1] studies the asymptotic behavior of solutions $y(x) \to 0$ $(x \to \infty)$ of the differential equation

$$y'' = \phi(x)y^\lambda, \quad \text{with } \lambda > 1 . \tag{1}$$

If ϕ is regularly varying with exponent $\sigma > -2$, notation $\phi \in \mathrm{RV}_\sigma$ (i.e., ϕ is measurable and eventually positive and $\phi(xy)/\phi(x) \to y^\sigma$ $(x \to \infty)$ for $y > 0$) and if $y(x)$ is a solution of (1) satisfying $y(x) \to 0$ $(x \to \infty)$, then

$$y(x) \sim \left[\frac{(1+\lambda+\sigma)(\sigma+2)}{(\lambda-1)^2}\right]^{1/(\lambda-1)} \{x^2\phi(x)\}^{-1/(\lambda-1)} \qquad (x \to \infty) . \tag{2}$$

The above result is generalized to the equation $y'' = f(x)\phi(y)$ in three papers by Marić and Tomić [5, 6, 7]. A related paper is Omey [8].

Here we present a simple proof of the original result using the following well-known approximation result on regularly varying functions:

Lemma (see [2, Theorem 17]). *Suppose $f \in \mathrm{RV}_\alpha$. Then there exist two functions $f_1 \sim f_2$ such that $f_1(t) \le f(t) \le f_2(t)$ for $t \ge t_0$ and such that the functions $\psi_i(t) := \log f_i(e^t)$ are C^∞ on a neighborhood of ∞ and satisfy*

$$\psi_i'(\tau) \to \alpha \qquad (\tau \to \infty)$$

and

$$\psi_i^{(n)}(\tau) \to 0 \qquad (\tau \to \infty), \ n \ge 2,$$

for $i = 1, 2$.

Theorem. *If y is a bounded positive solution of the differential equation $y'' = \phi(x)y^\lambda$ with $\phi \in \mathrm{RV}_\sigma$, $\sigma > -2$, and $\lambda > 1$ constant, then*

$$y \in \mathrm{RV}_{-(\sigma+2)/(\lambda-1)} .$$

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Proof. Substitution of $u = y^{1-\lambda}$ and $v(x) = \log u(e^x)$ shows that v satisfies the equation

$$v'' - v' - \beta v'^2 = -e^{\psi - v}, \tag{3}$$

where $\psi = \psi(x) \log\{(\lambda - 1)e^{2x}\phi(e^x)\}$ and $\beta = (\lambda - 1)^{-1} > 0$.

By the lemma, there exists ψ_1 with $\psi - \psi_1 \to 0$, $\psi_1' \to \sigma + 2$, $\psi_1'' \to 0$, and $\psi_1 \le \psi$ for x sufficiently large. Substituting $v = \psi_1 + c$ in (3) now gives

$$c'' - \gamma c' - \beta c'^2 = -(1 + o(1))e^{-c} + (\sigma + 2)(1 + \beta\sigma + 2\beta) + o(1), \tag{4}$$

with $\gamma := \gamma(x) \to 2\beta(\sigma + 2) + 1 \quad (x \to \infty)$. We claim that $c = c(x)$ tends to a finite limit as $x \to \infty$.

The following three cases are possible:

(i) $c' > 0$ *for* $x > x_0$. Then c is ultimately increasing; i.e., $\lim c(x) \le \infty$ exists. If $c(x) \to \infty$, then by (4), for x sufficiently large, $\delta' > \gamma\delta + \beta\delta^2 > \frac{1}{2}\delta$, where $\delta := c'$. This implies $\delta \to \infty$ and by (4), $(-1/\delta)' = \delta'/\delta^2 \to \beta$; hence, $-1/c' = -1/\delta \sim \beta x \quad (x \to \infty)$. This contradicts the assumption $c' > 0$.

(ii) *There is a sequence* $x_k \to \infty \quad (k \to \infty)$ *with* $c'(x_k) = 0$. Assume x_k is the sequence of all consecutive zeros of c. If $c''(x_k) < 0$ (c attains its maximum in x_k) and $\varepsilon > 0$ arbitrary, then $c(x_k) < -\log\{(\sigma+2)(1+\beta\sigma+2\beta)\}$ for large k, by (4). Similarly, if $c''(x_k) > 0$ we find $c(x_k) > -\log\{(\sigma+2)(1+\beta\sigma+2\beta)\}$; hence, a contradiction.

(iii) $c' < 0$ *for* $x > x_0$. Then c is ultimately decreasing. If $c(x) \to \infty$ $(x \to \infty)$, then, since $\psi_1 \le \psi$, we have, using (3),

$$-v'' + v' + \beta v'^2 = e^{\psi - v} \ge e^{\psi_1 - v} = e^{-c}.$$

Hence there exists a sequence $x_n \to \infty$ $(n \to \infty)$ such that $v'(x_n) \to \pm\infty$. If $v'(x_n) \to +\infty$, then $c'(x_n) \to +\infty$; hence, a contradiction. The case $v'(x_n) \to -\infty$ implies $u'(\exp x_n) < 0$; hence, $y'(\exp x_n) > 0$ for n sufficiently large. Since $y'' > 0$, this contradicts the boundedness of y.

This finishes the proof, since $\psi - v \to$ constant implies $x^2\phi(x) \sim cy^{1-\lambda}(x)$; hence, y is regularly varying.

Remark. The conclusion $y \in \mathrm{RV}_{-(\sigma+2)/(\lambda-1)}$ implies that $y \to 0$. Moreover, y'' is regularly varying as the product of two regularly varying functions. Application of Karamata's theorem (see, e.g., [3, 4]) then gives $x^2 y'' \sim c_0 y$ with $c_0 = (\sigma + 2)(\sigma + 1 + \lambda)/(\lambda - 1)^2$. Substituting this in (1) gives (2).

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