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ARRAY CONVERGENCE OF FUNCTIONS OF THE FIRST BAIRE CLASS

HELMUT KNAUST

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ABSTRACT. We show that every array $(x(i, j)\colon 1 \leq i < j < \infty)$ of elements in a pointwise compact subset of the Baire-1 functions on a Polish space, whose iterated pointwise limit $\lim_i \lim_j x(i, j)$ exists, is converging Ramsey-uniformly. An array $(x(i, j)_{i<j})$ in a Hausdorff space $\mathscr{T}$ is said to converge Ramsey-uniformly to some x in $\mathscr{T}$, if every subsequence of the positive integers has a further subsequence (m_i) such that every open neighborhood U of x in $\mathscr{T}$ contains all elements $x(m_i, m_j)$ with $i < j$ except for finitely many i.

1. INTRODUCTION

It is a well-known consequence of Ramsey's theorem that every array $(a_{ij})_{i<j}$ of real numbers with $\lim_i \lim_j a_{ij} = a$ for some $a \in \mathbb{R}$ has the following property: There is a subsequence (m_i) so that for all $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that $|a_{m_i m_j} - a| < \varepsilon$ for all $n < m_i < m_j$. This result generalizes easily to Hausdorff spaces which satisfy the first countability axiom.

The purpose of our note is to show that a corresponding result holds for the space of functions of the first Baire class $\mathscr{B}_1(\Omega)$ on a Polish space Ω, given the topology of pointwise convergence.

Let us say that an array $(x(i, j)\colon 1 \leq i < j < \infty)$ of elements in a Hausdorff space $\mathscr{T}$ converges *Ramsey-uniformly* to some $x \in \mathscr{T}$, if every subsequence of $\mathbb{N}$ has a further subsequence (m_i) such that for every open neighborhood U of x in $\mathscr{T}$ there is an $n \in \mathbb{N}$ so that $x(m_i, m_j) \in U$ for all $n < m_i < m_j$.

With this notation we can state our main result as follows:

Theorem 1. *Let Ω be a Polish space and let K be a pointwise compact subset of $\mathscr{B}_1(\Omega)$. If x and $(x(i, j))_{i<j}$ are elements in K with $\lim_i \lim_j x(i, j) = x$, then $(x(i, j))$ converges Ramsey-uniformly to x.*

A topological space Ω is *Polish*, if it is homeomorphic to a complete separable metric space. A real-valued function is *of the first Baire class on* Ω, if it is the pointwise limit of a sequence of continuous functions on Ω.

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It is a fundamental result of Bourgain, Fremlin, and Talagrand [2] that $\mathscr{B}_1(\Omega)$ is an angelic space, if Ω is Polish. A Hausdorff space $\mathscr{T}$ is *angelic*, if for every relatively compact subset A of $\mathscr{T}$ each point in the closure of A is the limit of a sequence in A and if relatively countably compact sets in $\mathscr{T}$ are relatively compact. In angelic spaces the notions of (relative) compactness, (relative) countable compactness, and (relative) sequential compactness coincide. Further basic results about angelic spaces can be found in [7].

Theorem 1 strengthens—in the case of functions of the first Baire class on a Polish space—a result of Boehme and Rosenfeld [1, Theorem 1], which we phrase for our purposes as follows:

Lemma 2. *Let $\mathscr{T}$ be an angelic space, and let x and $(x(i, j))_{i<j}$ be elements in a compact subset of $\mathscr{T}$ with $\lim_i \lim_j x(i, j) = x$. Then there is a subsequence (m_i) of $\mathbb{N}$ with $\lim_k x(m_{2k-1}, m_{2k}) = x$.*

Lemma 2 was also obtained independently, in the $\mathscr{B}_1(\Omega)$-setting, by Rosenthal [8, Theorem 3.6].

From Theorem 1 and a result by Odell and Rosenthal [6] we obtain the following Banach space corollary:

Corollary 3. *Let X be a separable Banach space not containing l_1. If x^{**} and $(x^{**}(i, j))_{i<j}$ are elements in a bounded subset of X^{**} with*

$$\omega^*\text{-}\lim_i \omega^*\text{-}\lim_j x^{**}(i, j) = x^{**},$$

then $(x^{**}(i, j))$ converges Ramsey-uniformly to x^{**} in the ω^*-topology.

The proof of Theorem 1 utilizes Lemma 2 to extract "nice" converging subsequences out of the given array $(x(i, j))$. We use Ramsey theory to produce the subarray for which one obtains Ramsey-uniform convergence.

If M is an infinite subset of $\mathbb{N}$, $\mathscr{P}^\infty(M)$ will denote the set of all infinite subsets of M. We give $\mathscr{P}^\infty(\mathbb{N})$ the topology, which is inherited by considering $\mathscr{P}^\infty(\mathbb{N})$ as a subspace of $\{0, 1\}^{\mathbb{N}}$ endowed with the product topology.

A subset $\mathscr{A} \subset \mathscr{P}^\infty(\mathbb{N})$ is called a *Ramsey set*, if for all $L \in \mathscr{P}^\infty(\mathbb{N})$ there is an $M \in \mathscr{P}^\infty(L)$ such that either $\mathscr{P}^\infty(M) \subset \mathscr{A}$ or $\mathscr{P}^\infty(M) \cap \mathscr{A} = \varnothing$. It is known that analytic (and coanalytic) subsets of $\mathscr{P}^\infty(\mathbb{N})$ are Ramsey sets [3, 9]. For a proof of this result, some history and more general results, see [5].

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2. Proof

Proof of Theorem 1. Let x and $x(i, j)$ with $1 \le i < j < \infty$ be elements in K such that $\lim_i \lim_j x(i, j) = x$. We let

$$\mathscr{A} = \{M = (m_i) \in \mathscr{P}^\infty(\mathbb{N})\colon\ (x(m_{2k-1}, m_{2k}))_{k=1}^\infty \text{ is pointwise convergent}\}.$$

Lemma 4. *$\mathscr{A}$ is coanalytic.*

We postpone the proof of the lemma and proceed with the proof of the theorem. Since $\mathscr{A}$ is coanalytic, $\mathscr{A}$ is a Ramsey set. Let $L \in \mathscr{P}^{\infty}(\mathbb{N})$. We can thus find $M = (m_i)_{i=1}^{\infty} \in \mathscr{P}^{\infty}(L)$ so that $\mathscr{P}^{\infty}(M) \subset \mathscr{A}$ or $\mathscr{P}^{\infty}(M) \cap \mathscr{A} = \varnothing$. Lemma 2 shows that the first alternative holds. Moreover, Lemma 2 asserts that $\lim_k x(m'_{2k-1}, m'_{2k}) = x$ for some $M' = (m'_i) \in \mathscr{P}^{\infty}(M)$.

Suppose now the conclusion of Theorem 1 fails for M'. Then there is an open neighborhood U of x and a subsequence $M'' \subset M'$ with

$$x(m''_{2k-1}, m''_{2k}) \notin U \quad \text{for all } k \in \mathbb{N}.$$

Since $M'' \in \mathscr{P}^{\infty}(M)$, we have $M'' \in \mathscr{A}$ and thus $\lim_k x(m''_{2k-1}, m''_{2k}) = y$ for some $y \in \mathscr{B}_1(\Omega)$. Note that $y \neq x$.

We now construct a subsequence $N = (n_i) \in \mathscr{P}^{\infty}(M)$ inductively as follows: Let $n_1 = m'_1$ and $n_2 = m'_2$. Once $n_1, n_2, \dots, n_{2k}$ have been chosen, we define n_{2k+1} and n_{2k+2} as follows: If k is odd, we choose an $l \in \mathbb{N}$ so that $m''_{2l-1} > n_{2k}$ and let $n_{2k+1} = m''_{2l-1}$, $n_{2k+2} = m''_{2l}$. If k is even, we can find an $l \in \mathbb{N}$ with $m'_{2l-1} > n_{2k}$ and then let $n_{2k+1} = m'_{2l-1}$, $n_{2k+2} = m'_{2l}$. On the one hand the sequence $(x(n_{2k-1}, n_{2k}))$ is pointwise convergent, on the other hand it contains two subsequences converging to x and y respectively. This yields a contradiction.

Proof of Lemma 4. The proof of Lemma 4 uses techniques similar to those employed in [10].

Let Y be the set of all real-valued arrays $(a(i, j))_{i<j}$, endowed with the topology of pointwise convergence. We set $Z = \mathscr{P}^{\infty}(\mathbb{N}) \times Y$ and denote by $\phi\colon \Omega \to Y$ the canonical map defined by $\phi(\omega) = (x(i, j)(\omega))_{i<j}$. Since ϕ is a Borel-measurable map and Ω is Polish, $\phi(\Omega)$ is analytic in Y (see [4, §38]). Consequently $Z_1 := \mathscr{P}^{\infty}(\mathbb{N}) \times \phi(\Omega)$ is analytic in Z.

We define a set $Z_2 \subset \mathscr{P}^{\infty}(\mathbb{N}) \times Y$ as follows:

$$Z_2 = \{(M, (a(i, j))) : (a(m_{2k-1}, m_{2k}))_{k=1}^{\infty} \text{ is not Cauchy}\}.$$

Observing that the set

$$Z_2^{l,N} := \{(M, (a(i, j))) : \text{ there are } k_1, k_2 > N \text{ with } |a(m_{2k_1-1}, m_{2k_1}) - a(m_{2k_2-1}, m_{2k_2})| > 2^{-l}\}$$

is open, and that

$$Z_2 = \bigcup_{l \in \mathbb{N}} \bigcap_{N \in \mathbb{N}} Z_2^{l,N},$$

we obtain that Z_2 is a $G_{\delta\sigma}$-set in Z.

Consequently $Z_1 \cap Z_2$ is analytic in Z. We let $P\colon Z \to \mathscr{P}^{\infty}(\mathbb{N})$ be the projection of Z onto its first coordinate. One can see easily that the complement of $\mathscr{A}$ is equal to $P(Z_1 \cap Z_2)$. Thus $\mathscr{P}^{\infty}(\mathbb{N}) \setminus \mathscr{A}$ is analytic in $\mathscr{P}^{\infty}(\mathbb{N})$ as the continuous image of an analytic set in Z (see [4, §38]).

Problem. Does Theorem 1 hold for arbitrary angelic spaces? Lemma 2 reduces this problem to the apparently open question, whether the set $\mathscr{A} \subset \mathscr{P}^{\infty}(\mathbb{N})$, defined at the beginning of the proof, is still a Ramsey set for arbitrary angelic spaces.

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74078