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A LOCAL CLASSIFICATION OF 2-TYPE SURFACES IN S^3

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ABSTRACT. The only spherical surfaces in E^4 that are either of 1-type or of 2-type are portions of ordinary spheres, minimal surfaces in S^3, and Riemannian products of two plane circles of different radii.

1. INTRODUCTION

Let M be a connected (not necessarily compact) surface of the four-dimensional Euclidean space E^4, endowed with the induced metric. Denote by Δ the Laplacian of M acting on smooth functions on M. This Laplacian can be extended in a natural way to E^4-valued smooth maps on M. A surface M of E^4 is said to be of k-type (see [2] for details) if the position vector x of M in E^4 admits the following spectral decomposition

$$x = x_0 + x_{i_1} + \cdots + x_{i_k},$$

where x_0 is a constant vector and x_{i_j} $(j = 1, \dots, k)$ are nonconstant E^4-valued maps on M such that

$$\Delta x_{i_j} = \lambda_{i_j} x_{i_j}, \qquad \lambda_{i_j} \in R,\ \lambda_{i_1} < \lambda_{i_2} < \cdots < \lambda_{i_k}.$$

In this article we consider surfaces of E^4, which lie in a hypersphere of E^4 and are at most of 2-type. This means that the position vector can be expressed in the following form

$$x = x_0 + x_1 + x_2, \qquad \Delta x_1 = \lambda_1 x_1, \qquad \Delta x_2 = \lambda_2 x_2,$$

where x_0 is a constant vector and $\lambda_1 = \lambda_2$ (1-type) or $\lambda_1 \neq \lambda_2$ (2-type).

B.-Y. Chen [2, 3] has proved that the products of two plane circles of different radii are the only compact mass-symmetric surfaces of S^3 which are of 2-type. Recently, M. Barros and O. Garay [1] showed the same result without the assumption that the surface was mass-symmetric.

In the present article we study the same problem without the assumption of

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compactness. More precisely, we prove the following local result:

Theorem 1. *Let M be a surface of a hypersphere of E^4. M is of 2-type if and only if it is a piece of a Riemannian product of two plane circles of different radii.*

A direct consequence is the following corollary:

Corollary. *The only connected complete 2-type surfaces of S^3 are the Riemannian products of two plane circles of different radii.*

Moreover, we prove the following theorem:

Theorem 2. *The only spherical surfaces of E^4 that are at most of 2-type are the following:*

(i) *portions of ordinary spheres,*
(ii) *portions of minimal surfaces of hyperspheres, and*
(iii) *portions of Riemannian products of two plane circles of different radii.*

2. Preliminaries

Consider a surface M of E^4 and denote by Δ the Laplacian of M associated with the induced metric. This Laplacian can be extended in a natural way to E^4-valued smooth maps on M. More precisely, let (X_1, X_2) be a local orthonormal frame field of M and v an E^4-valued map on M. We define

$$\Delta v = \sum_{i=1}^{2}(\overline{\nabla}_{X_i}\overline{\nabla}_{X_i}v - \overline{\nabla}_{\nabla_{X_i}X_i}v), \tag{2.1}$$

where $\overline{\nabla}$ denotes the usual Riemannian connection of E^4 and ∇ the induced connection on M.

We suppose that the position vector of M with respect to the origin O of E^4 can be written as follows

$$x = x_0 + x_1 + x_2, \qquad \Delta x_1 = \lambda_1 x_1, \qquad \Delta x_2 = \lambda_2 x_2, \tag{2.2}$$

where x_0 is a constant vector in E^4 and x_1, x_2 are nonconstant E^4-valued maps on M. If, in addition, M lies in a hypersphere S_R^3 of radius R centered at b, then we have

$$y = a + x_1 + x_2, \tag{2.3}$$

where y denotes the position vector of M with respect to the center of S_R^3 and $a = x_0 - b$ is a constant vector in E^4.

Hereafter, we denote by H the mean curvature vector of M in E^4. Since $\Delta y = 2H$ we have

$$2H = \lambda_1 x_1 + \lambda_2 x_2 \quad \text{and} \quad 2\Delta H = \lambda_1^2 x_1 + \lambda_2^2 x_2,$$

and thus

$$\Delta H = (\lambda_1 + \lambda_2)H - \frac{\lambda_1\lambda_2}{2}(y - a). \tag{2.4}$$

Obviously, the unit vector field $e_3 = y/R$ is normal to M and to S_R^3. Let e_4 be the unit vector field which, together with e_3, builds a basis of the normal space of M in E^4. It is evident that e_4 is the unit vector field, normal to M in S_R^3. Now, we decompose the vector a at each point $P \in M$ in a component a_{T} tangent to M, and a component normal to M; that is

$$a = a_{\mathrm{T}} + \langle a, e_3\rangle e_3 + \langle a, e_4\rangle e_4, \tag{2.5}$$

where $\langle\ ,\ \rangle$ stands for the usual inner product of E^4. This decomposition takes place in the tangent space of E^4 at the point P, where a is identified, by parallel translation in E^4, with a vector in that tangent space. Evidently we have

$$H = -\frac{1}{R}e_3 + \frac{\operatorname{tr} A_4}{2}e_4, \tag{2.6}$$

where A_4 is the Weingarten map of M associated with e_4.

Replacing (2.5) and (2.6) in (2.4) we obtain

$$\begin{aligned}\Delta H = {} & \frac{\lambda_1\lambda_2}{2}a_{\mathrm{T}} + \left(-\frac{\lambda_1+\lambda_2}{R} - \frac{\lambda_1\lambda_2}{2}R + \frac{\lambda_1\lambda_2}{2}\langle a, e_3\rangle\right)e_3 \\ & + \left(\frac{\lambda_1\lambda_2}{2}\langle a, e_4\rangle + \frac{\lambda_1+\lambda_2}{2}\operatorname{tr} A_4\right)e_4.\end{aligned} \tag{2.7}$$

Moreover, applying (2.1) to H we get, by easy computations, that

$$\begin{aligned}\Delta H = {} & -A_4(\operatorname{grad}\operatorname{tr} A_4) - \frac{\operatorname{tr} A_4}{2}\operatorname{grad}\operatorname{tr} A_4 \\ & + \left(\frac{2}{R^3} + \frac{(\operatorname{tr} A_4)^2}{2R}\right)e_3 + \left(\Delta\left(\frac{\operatorname{tr} A_4}{2}\right) - \frac{\operatorname{tr} A_4}{R^2} - \frac{\operatorname{tr} A_4}{2}\operatorname{tr} A_4^2\right)e_4.\end{aligned} \tag{2.8}$$

Comparing (2.7) with (2.8) we obtain the following equations

$$A_4(\operatorname{grad}\operatorname{tr} A_4) + \frac{\operatorname{tr} A_4}{2}\operatorname{grad}\operatorname{tr} A_4 = -\frac{\lambda_1\lambda_2}{2}a_{\mathrm{T}}, \tag{2.9}$$

$$\frac{2}{R^3} + \frac{(\operatorname{tr} A_4)^2}{2R} = -\frac{\lambda_1+\lambda_2}{R} - \frac{\lambda_1\lambda_2}{2}R + \frac{\lambda_1\lambda_2}{2}\langle a, e_3\rangle, \tag{2.10}$$

$$\Delta\left(\frac{\operatorname{tr} A_4}{2}\right) - \frac{\operatorname{tr} A_4}{R^2} - \frac{\operatorname{tr} A_4}{2}\operatorname{tr} A_4^2 = \frac{\lambda_1\lambda_2}{2}\langle a, e_4\rangle + \frac{\lambda_1+\lambda_2}{2}\operatorname{tr} A_4. \tag{2.11}$$

We differentiate covariantly equation (2.5) in the direction of an arbitrary tangent vector X of M. Using the Gauss formula for S_R^3 in E^4 and for M in S_R^3, as well as the Weingarten formula for the direction e_4 in S_R^3, we compute

$$\begin{aligned}0 = \overline{\nabla}_X a = {} & \nabla_X a_{\mathrm{T}} - \frac{\langle X, a_{\mathrm{T}}\rangle}{R}e_3 + \langle A_4 a_{\mathrm{T}}, X\rangle e_4 \\ & + X\langle a, e_3\rangle e_3 + \frac{\langle a, e_3\rangle}{R}X + X\langle a, e_4\rangle e_4 - \langle a, e_4\rangle A_4 X.\end{aligned}$$

Taking the tangential and the normal component of both sides of this equation we obtain

$$\nabla_X a_T + \frac{\langle a, e_3\rangle}{R} X - \langle a, e_4\rangle A_4 X = 0,$$
$$X\langle a, e_3\rangle = \frac{\langle X, a_T\rangle}{R},$$
$$X\langle a, e_4\rangle = -\langle A_4 a_T, X\rangle.$$

The last two equations become

$$\operatorname{grad}\langle a, e_3\rangle = \frac{a_T}{R}, \tag{2.12}$$

$$\operatorname{grad}\langle a, e_4\rangle = -A_4 a_T. \tag{2.13}$$

By taking the gradient of both sides of (2.10) and using (2.12) we get

$$\operatorname{tr} A_4 \operatorname{grad} \operatorname{tr} A_4 = \frac{\lambda_1\lambda_2}{2} a_T, \tag{2.14}$$

from which (2.9) becomes

$$A_4 \operatorname{grad} \operatorname{tr} A_4 = -\tfrac{3}{2} \operatorname{tr} A_4 \operatorname{grad} \operatorname{tr} A_4. \tag{2.15}$$

Finally, combining (2.13)–(2.15) we find

$$\operatorname{grad}(\lambda_1\lambda_2\langle a, e_4\rangle) = \operatorname{grad}(\operatorname{tr} A_4)^3,$$

from which we obtain

$$\lambda_1\lambda_2\langle a, e_4\rangle = (\operatorname{tr} A_4)^3 + c, \tag{2.16}$$

where c is a constant.

3. Proofs

At first we need the following local lemma that is the crucial point in our proofs.

Lemma. *Let M be a surface of a 3-sphere S^3_R in E^4, which is of 2-type. Then M has constant mean curvature in S^3_R. In particular, the mean curvature vector H of M in E^4 is parallel in the normal bundle of M.*

Proof. It is enough to prove that $\operatorname{grad} \operatorname{tr} A_4$ is zero everywhere on M. On the contrary, we suppose that $\operatorname{grad} \operatorname{tr} A_4$ is nonzero at $P_0 \in M$. Then, there exists an open neighborhood V of P_0 in M, where $\operatorname{grad} \operatorname{tr} A_4$ is nowhere zero. Hereafter we work in V. Equation (2.15) implies that $\operatorname{grad} \operatorname{tr} A_4$ is an eigenvector of A_4 with corresponding eigenvalue $-\frac{3}{2} \operatorname{tr} A_4$ and thus the other eigenvalue of A_4 is $\frac{5}{2} \operatorname{tr} A_4$.

Choosing $e_1 = \operatorname{grad} \operatorname{tr} A_4 / |\operatorname{grad} \operatorname{tr} A_4|$ and e_2 the oriented complement of e_1 in the tangent space of M, we get a moving tangent frame (e_1, e_2). We denote by (w_1, w_2) the dual frame and by w_{12} the connection form, which is defined by $w_{12}(X) = \langle \nabla_X e_1, e_2 \rangle$, where X is a tangent vector of M. The Weingarten map A_4 has the form, with respect to the basis e_1, e_2:

$$A_4 \sim \begin{pmatrix} -\frac{3}{2}\operatorname{tr} A_4 & 0 \\ 0 & \frac{5}{2}\operatorname{tr} A_4 \end{pmatrix}.$$

Therefore the Gauss curvature of M, restricted on V, is given by

$$K = \frac{1}{R^2} - \frac{15}{4}(\operatorname{tr} A_4)^2. \tag{3.1}$$

Now, the Codazzi equation $(\nabla_{e_1} A_4)e_2 = (\nabla_{e_2} A_4)e_1$ implies

$$\tfrac{5}{8}e_1(\operatorname{tr} A_4) = -\operatorname{tr} A_4 w_{12}(e_2), \qquad \operatorname{tr} A_4 w_{12}(e_1) = 0, \tag{3.2}$$

since $e_2(\operatorname{tr} A_4) = 0$, by our choice. From the first equation of (3.2) we conclude that $\operatorname{tr} A_4$ is nonzero on V and thus from the second of (3.2) we obtain $w_{12}(e_1) = 0$. So, $\nabla_{e_1} e_1 = 0$, which means that the integral curves of e_1 are geodesics of M. Let $\gamma(s)$ be the integral curve of e_1, which emanates from P_0. Moreover, from (3.2) we have

$$w_{12} = -\frac{5}{8}\frac{e_1(\operatorname{tr} A_4)}{\operatorname{tr} A_4} w_2. \tag{3.3}$$

Taking the exterior differentiation of (3.3) and applying the structure equations we obtain

$$K = \frac{5}{8}\frac{e_1(e_1(\operatorname{tr} A_4))}{\operatorname{tr} A_4} - \frac{65}{64}\left(\frac{e_1(\operatorname{tr} A_4)}{\operatorname{tr} A_4}\right)^2. \tag{3.4}$$

Setting $h(s) = \operatorname{tr} A_4(\gamma(s))$ and comparing (3.1) with (3.4) we get the following second order differential equation

$$hh'' - \frac{13}{8}(h')^2 = -6h^4 + \frac{8}{5R^2}h^2, \tag{3.5}$$

where prime denotes derivative with respect to s. From the last equation we find

$$(h')^2 = c_1 h^{13/4} - 16h^4 - \frac{64}{25R^2}h^2, \tag{3.6}$$

where c_1 is a constant.

On the other hand equation (2.11), taking into account (2.16) and the form of A_4, gives along $\gamma(s)$

$$hh'' - \frac{5}{8}(h')^2 = \frac{19}{2}h^4 + \left(\frac{2}{R^2} + \lambda_1 + \lambda_2\right)h^2 + ch. \tag{3.7}$$

Therefore, from (3.5) and (3.7) we obtain

$$(h')^2 = \frac{31}{2}h^4 + \left(\frac{2}{5R^2} + \lambda_1 + \lambda_2\right)h^2 + ch. \tag{3.8}$$

Comparing (3.6) with (3.8) we conclude that $h(s)$ is constant; that is $\operatorname{tr} A_4$ is constant along $\gamma(s)$ and thus $\operatorname{grad}\operatorname{tr} A_4$ is zero at P_0. This contradicts our assumption and completes the proof of the lemma.

Proof of Theorem 1. By virtue of the lemma we have that $\operatorname{tr} A_4$ is constant on M. Moreover, M has mean curvature vector in E^4 parallel in the normal bundle. On the other hand, from (2.11) we conclude that the length of the second fundamental form is also constant. Hence, M has constant Gauss curvature. According to a result of B.-Y. Chen and G. D. Ludden [4] M must be a piece of a Riemannian product of two plane circles of different radii, since the ordinary spheres are of 1-type. The converse is satisfied by computations.

Proof of Theorem 2. It is enough to consider the two cases: (i) $\lambda_1 = \lambda_2 = 0$ and (ii) $\lambda_1 = \lambda_2 \neq 0$. In the first case M is minimal in E^4, which is impossible, since M is also spherical. In the second case, from (2.3) we obtain $\Delta(y-a) = \lambda(y-a)$. Therefore, by the well-known Takahashi Theorem [5], it follows that M is minimal in a 3-sphere $S^3_{R_0}$ with center $x_0 = b + a$ and radius $R_0 = \sqrt{2/|\lambda|}$. So, M is a piece of an ordinary sphere $(a \neq 0)$ or a piece of a minimal surface of S^3_R $(a = 0)$.

References

1. M. Barros and O. Garay, *2-type surfaces in S^3*, Geometriae Dedicata **24** (1987), 329–336.
2. B.-Y. Chen, *Total mean curvature and submanifolds of finite type*, World Scientific, Singapore and New Jersey, 1984.
3. ——, *2-type submanifolds and their applications*, Chinese J. Math. **14** (1986), 1–14.
4. B.-Y. Chen and G. D. Ludden, *Surfaces with mean curvature vector parallel in the normal bundle*, Nagoya Math. J. **47** (1972), 161–167.
5. T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385.

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