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NEW COMPLETE GENUS ZERO MINIMAL SURFACES WITH EMBEDDED PARALLEL ENDS

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ABSTRACT. We construct genus zero complete minimal surfaces of finite total curvature embedded outside a compact set of $\mathbb{R}^3$.

INTRODUCTION

Research about embedded complete minimal surfaces of finite total curvature in $\mathbb{R}^3$ has experienced a great development throughout the last years.

Some basic properties of these surfaces have been studied by Jorge and Meeks [1]. Concretely, they showed that the ends of such surfaces are parallel, and the normal vectors at these ends are distributed in a symmetric way. In the genus zero case, Meeks and Jorge [1] proved also that there does not exist a complete minimal surface (not necessarily embedded) with 3, 4, or 5 embedded ends, satisfying the above geometric properties.

Later, Peng [5] called this kind of minimal surfaces pseudo-embedded, and found examples with genus zero and 6 ends. Liang in [2] constructed examples with k ends, $k \geq 6$.

In this paper we describe a family of genus zero pseudo-embedded minimal surfaces, which we will call *Catenoid-type surfaces*, with interesting geometric properties. A Catenoid-type surface has only two Catenoid ends with opposite normal vectors. Moreover, it has an even number of ends and its Gauss map omits two points of the sphere.

Of course, the Catenoid is a trivial example of Catenoid-type surface, and there does not exist a Catenoid-type surface with 3, 4, or 5 ends (see [1]).

Genus zero pseudo-embedded minimal surfaces can be constructed by solving a general equation system (see [5]). Hence Catenoid-type surfaces are solutions of these equations. Essentially, Peng [5] and Liang [2] have proved that there exist solutions for this system in the general case (except for surfaces with 3, 4, or 5 ends), although they did not make a careful study of the geometric properties of their examples.

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Recently F. J. López and A. Ros [3] have proved that the Catenoid and the Plane are the only genus zero embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$.

In Corollary 2 we prove that there exist nontrivial Catenoid-type surfaces embedded outside a compact set of $\mathbb{R}^3$. This fact shows that even in the next simplest case, the result in [3] must fail, and these surfaces are critical to the proof of the above theorem.

On the other hand, S. Montiel and A. Ros (see [4]) have proved that the Index of a genus zero minimal surface of finite total curvature is equal to $2d-1$, where d is the degree of the Gauss map, except on a critical family of surfaces.

Among these critical surfaces are pseudo-embedded minimal surfaces. Then, to compute the Index of our explicit examples could be an interesting problem.

Preliminaries

Let $x\colon M \longrightarrow \mathbb{R}^3$ be an orientable nonflat complete minimal surface of finite total curvature in the Euclidean space $\mathbb{R}^3$.

Denote by ω, g the holomorphic 1-form and the meromorphic function on M determined by the Weierstrass representation of x, respectively (see [6]). Modulo natural identifications, g is the Gauss map of M.

It is well known that

$$x = \operatorname{Real} \int (\phi_1, \phi_2, \phi_3), \tag{1}$$

where $\phi_1 = \omega(1-g^2)/2$, $\phi_2 = i\omega(1+g^2)/2$, and $\phi_3 = \omega g$.

Osserman [6] proved that M is conformally equivalent to $\overline{M}-\{P_1, \dots, P_r\}$, where $\overline{M}$ is a compact Riemann surface, $\{P_1, \dots, P_r\}$ are points of $\overline{M}$ and ω, g extend meromorphically to $\overline{M}$. So, we can define the normal vector at each end of M.

By definition, we put $\operatorname{Genus}(M) = \operatorname{Genus}(\overline{M})$.

An end P_i is embedded if and only if (see [1])

$$\operatorname{Max}\left\{O(\phi_j, P_i),\ j = 1, 2, 3\right\} = 2, \tag{2}$$

where $O(\phi_j, P_i)$ is the order of the pole that ϕ_j has at P_i.

When P_i is a regular point of g, x is asymptotic to a Catenoid at P_i, and the end is called a Catenoid end. If P_i is a branch of g, x is asymptotic to a plane and the end is called a planar end.

Definition 1. The surface M is of Catenoid type if and only if:

(1) $\operatorname{Genus}(M) = 0$
(2) The ends of M are embedded and parallel, that is, there exists $a \in \mathbb{S}^2$ such that $g\{P_1, \dots, P_r\} \subset \{a, -a\}$.
(3) M has only two Catenoid ends whose normal vectors point at opposite directions.

Since finite-total-curvature surfaces are parabolic, and Catenoid-type surfaces have parallel ends, the Catenoid ends go to infinity in opposite directions. Condition (3) in the above definition means that the Catenoid ends have opposite values of g.

The Catenoid is a trivial example of a surface in this family. In the following we will exclude it in all our results.

Construction of examples

We first state the following proposition:

Proposition 1. *Let M be a Catenoid-type surface. Then there exist $A, B \in \mathbb{C} - \{0\}$, $AB \in \mathbb{R}$, and $P(z) = \prod_{j=1}^{k}(z - a_j)$, $Q(z) = \prod_{j=1}^{k}(z - b_j)$, $k \geq 2$, polynomial functions such that:*

M is conformally equivalent to $\mathbb{C} - \{0, a_1, \dots, a_k, b_1, \dots, b_k\}$,

$$P'(a_j), Q'(b_j), P(b_j), Q(a_j) \neq 0 \quad \forall j = 1, \dots, k, \tag{3}$$

$$\begin{cases} zPQ'' + zQP'' + 2QP' - 2P'Q' = 0, \\ P'(0) = Q'(0) = 0, \end{cases} \tag{4}$$

and up previous rotation in $\mathbb{R}^3$ and reparametrization in $\overline{\mathbb{C}}$:

$$g = Bz\frac{P^2(z)}{Q^2(z)}, \qquad \omega = A\frac{Q^2(z)}{z^2P^2(z)}dz. \tag{5}$$

Moreover, given $A, B \in \mathbb{C} - \{0\}$, $AB \in \mathbb{R}$, and P, Q polynomials verifying (3) and (4), if we take $M = \mathbb{C} - \{0, a_1, \dots, a_k, b_1, \dots, b_k\}$ and g, ω like in (5), the associated minimal surface via the Weierstrass representation is well defined and of Catenoid type.

Proof. Suppose that M is of Catenoid type. We can assume, up rotation in $\mathbb{R}^3$ and reparametrization in $\overline{\mathbb{C}}$, that the ends of M are poles and zeros of g and that $0, \infty$ are the two Catenoid ends of M.

By (2) and Definition 1, M is conformally equivalent to $\mathbb{C} - \{0, a_1, \dots, a_k, b_1, \dots, b_s\}$ and

$$g = Bz\frac{\prod_{i=1}^{k}(z - a_i)^{n_i}\prod_{i=k+1}^{t}(z - a_i)^{n_i}}{\prod_{i=1}^{s}(z - b_1)^{m_i}\prod_{i=s+1}^{u}(z - b_i)^{m_i}},$$

$$\omega = A\frac{\prod_{i=1}^{s}(x - b_i)^{2m_i - 2}\prod_{i=s+1}^{u}(z - b_i)^{2m_i}}{z^2\prod_{i=1}^{k}(z - a_i)^2}dz,$$

where $A, B \in \mathbb{C} - \{0\}$, a_i, $i = k+1, \dots, t$, b_i, $i = s+1, \dots, u$ are the zeros and poles of g not ends of M, and m_i, n_i are nonnegative integer numbers satisfying:

(6)

$$\sum_{i=1}^{t} n_i = \sum_{i=1}^{u} m_i = s + k \qquad n_i \geq 2,\ i = 1, \dots, k, \quad m_i \geq 2,\ i = 1, \dots, s,$$

The last identity follows from the fact that g has a simple pole at ∞ and ω is regular at this point (see (2)).

Using that $a_1, \dots, a_k, b_1, \dots, b_s$ are branch points of g, we deduce that

$$\sum_{i=1}^{u} m_i \geq 2s, \qquad \sum_{i=1}^{t} n_i \geq 2k. \tag{7}$$

Therefore, by (6) and (7)

$$s = k \quad \text{and} \quad \sum_{i=1}^{t} n_i = \sum_{i=1}^{u} m_i = 2k = 2s.$$

So, $m_i = n_i = 2$, $i = 1, \dots, k = s$, $m_i = n_i = 0$, $i > k = s$.
Then we can write $M \equiv \mathbb{C} - \{a_1, \dots, a_k, b_1, \dots, b_k\}$. Now define

$$P(z) = \prod_{i=1}^{k}(z - a_i), \qquad Q(z) = \prod_{i=1}^{k}(z - b_i).$$

Observe that these polynomials verify (3), and

$$g(z) = Bz\prod_{i=1}^{k}\left(\frac{P(z)}{Q(z)}\right)^2, \qquad \omega = \frac{A}{z^2}\prod_{i=1}^{k}\left(\frac{Q(z)}{P(z)}\right)^2 dz. \tag{8}$$

Using (1)

$$\phi_1 = \frac{1}{2}\left(A\frac{Q^2}{z^2P^2} - AB^2\frac{P^2}{Q^2}\right)dz, \qquad \phi_2 = \frac{i}{2}\left(A\frac{Q^2}{z^2P^2} + AB^2\frac{P^2}{Q^2}\right)dz,$$

$$\phi_3 = \frac{AB}{z}dz.$$

Since the 1-forms ϕ_i, $i = 1, 2, 3$ have no real periods on M, $AB \in \mathbb{R}$ and $(Q^2/P^2z^2)\,dz$, $(P^2/Q^2)\,dz$ are exact 1-forms.

On the other hand, it is easy to see that

$$\frac{Q^2}{z^2P^2}dz \text{ is exact } \Leftrightarrow \left(\frac{zP}{Q}\right)''(a_i) = 0, \qquad i = 1, \dots, k,$$

$$\left(\frac{P}{Q}\right)'(0) = 0 \Leftrightarrow \left(zP''Q + 2QP' - 2zQ'P'\right)(a_i) = 0, \qquad i = 1, \dots, k$$

$$\left(\frac{P}{Q}\right)'(0) = 0,$$

and analogously:

$$\frac{P^2}{Q^2}dz \text{ is exact } \Leftrightarrow \left(PQ'' - 2P'Q'\right)(b_i) = 0, \qquad i = 1, \dots, k.$$

Therefore $(Q^2/z^2P^2)\,dz$, $(P^2/Q^2)\,dz$ are exacts $\Leftrightarrow zPQ'' + zQP'' + 2QP' - 2zQ'P'$ vanishes at $a_1, \dots, a_k, b_1, \dots, b_k$ and $(P/Q)'(0) = 0$, and then (4).

Conversely, take $A, B \in \mathbb{C} - \{0\}$ and P, Q are polynomial functions of degree $k \geq 2$ satisfying (3) and (4).

If we define $M \equiv \mathbb{C} - \{0, a_1, \dots, a_k, b_1, \dots, b_k\}$ and g, ω as in (5), the 1-forms ϕ_i, $i = 1, 2, 3$ have no real periods and the minimal surface constructed via the Weierstrass representation is of Catenoid type. Q.E.D.

Corollary 1. *There exist minimal surfaces of Catenoid type with* $2k+2$ *ends, for each* $k \in \mathbb{Z}$, $k \geq 2$.

Proof. Fix $A, B \in \mathbb{C} - \{0\}$ such that $AB \in \mathbb{R}$. For each $k \geq 2$, take

$$P(z) = z^k - \left(\frac{k+1}{k-1}\right)\lambda, \qquad Q(z) = z^k + \lambda, \tag{9}$$

where $\lambda \in \mathbb{C} - \{0\}$.

For $k \geq 5$, also take

$$\begin{aligned} P(z) &= z^k - \frac{\lambda}{\mu}\left(\frac{k}{2r-k}\right)^2\left(\frac{k-r+1}{k-r-1}\right)z^r - \lambda\mu\left(\frac{r+1}{r-1}\right)z^{k-r} \\ &\quad + \frac{(r+1+(k-r+1)}{(r-1)(k-r-1)}\lambda^2, \\ Q(z) &= z^k + \frac{\lambda}{\mu}\left(\frac{k}{2r-k}\right)^2 z^r + \lambda\mu z^{k-r} + \lambda^2, \end{aligned} \tag{10}$$

where $\lambda, \mu \in \mathbb{C} - \{0\}$, $r \geq 2$, $r \neq k/2$.

By a long but straightforward computation, the polynomial functions in (9) and (10) satisfy (4) and give us minimal surfaces of Catenoid type provided that (3) is true. Q.E.D.

For a better understanding of Catenoid-type surfaces, we will show that some of them have all the planar ends at different heights. This nontrivial distribution of the ends in $\mathbb{R}^3$ is a necessary condition for embeddedness but, of course, not sufficient.

Lemma 1. *Take* k, r *positive integers such that* $k \geq 5$, $2 \leq r \leq k-2$, $r \neq k/2$, g.c.f. $(r, k-r) = 1$ *and fix* $\lambda \in \mathbb{C} - \{0\}$.

Then for almost all $\mu \in \mathbb{C} - \{0\}$, *the polynomial functions* P, Q *given by* (10) *satisfy*:

$$|a_i| \neq |a_j|, \qquad |b_i| \neq |b_j|, \quad \forall i, j, \ i \neq j, \ |b_i| \neq |a_j| \ \forall i, j. \tag{11}$$

Proof. First, observe that $\mu P(z, \mu)$, $\mu Q(z, \mu) \in \mathfrak{p}[z, \mu]$ are irreducible polynomials.

Therefore

$$\begin{aligned} M_1 &= \{(z, \mu)/z, \mu \in \overline{\mathbb{C}}, \ \mu P(z, \mu) = 0\}, \\ M_2 &= \{(z, \mu)/z, \mu \in \overline{\mathbb{C}}, \ \mu Q(z, \mu) = 0\} \end{aligned}$$

are connected Riemann surfaces.

Write by z_i, $i = 1, 2$ the corresponding projection on M_i, $i = 1, 2$, and note that $z_i^{-1}(\mu) = \left\{(z_i^1(\mu), \mu), \dots, (z_i^k(\mu), \mu)\right\}$, $i = 1, 2$ has k points counting multiplicities, for each $\mu \in \overline{\mathbb{C}}$.

Choose an open subset $U \subset \overline{\mathbb{C}}$ where z_1^h, $h = 1, \dots, k$ are well defined.

Suppose that $|z_1^i(\mu)| = |z_1^j(\mu)|$ for some $i, j \in \{1, \dots, k\}$, $i \neq j$, $\forall \mu \in U$. Then $z_1^i(\mu) = \theta z_1^j(\mu)$ on U, where $\theta \in \mathbb{C}$, $|\theta| = 1$.

Moreover, $\theta \neq 1$ because $\mu P(z, \mu)$ and $(\partial/\partial z(\mu P(z, \mu)))$ do not have any nontrivial common factor.

Define now $S(z, \mu) = \mu P(z, \mu) - \theta^{-k} P(\theta z, \mu)$.

Since g.c.f. $(r, k-r) = 1$ and $\theta \neq 1$, it is clear that $S(z, \mu) \neq 0$.

On the other hand, by analytic continuation along M_1, $S(z, \mu) = 0$ for each $(z, \mu) \in M_1$, and therefore, $S(z, \mu)$ and $P(z, \mu)$ have a nontrivial common factor, a contradiction.

Then it is easy to deduce that $|z_1^i(\mu)| \neq |z_1^j(\mu)|$ for almost all $\mu \in \overline{\mathbb{C}}$.

Analogously, $|z_1^i(\mu)| \neq |z_2^j(\mu)|$ for almost all $\mu \in \overline{\mathbb{C}}$, $i \neq j$.

Take now $U \subset \overline{\mathbb{C}}$ an open subset where z_i^h, $i = 1, 2$ and $h = 1, \dots, k$, are well defined.

If $|z_1^i(\mu)| = |z_2^j(\mu)|$ on U for some $i, j \in \{1, \dots, k\}$, then $z_1^i(\mu) = \theta z_2^j(\mu)$, where $\theta \in \mathbb{C}$, $|\theta| = 1$.

By analytic continuation along M_1, if $(z, \mu) \in M_2$ then $(\theta z, \mu) \in M_1$, and therefore $\mu Q(z, \mu) = \theta^{-k} \mu P(\theta z, \mu)$, a contradiction.

We can conclude that $|z_1^i(\mu)| \neq |z_2^j(\mu)|$ for almost all $\mu \in \overline{\mathbb{C}}$, $\forall i, j$. Q.E.D.

Corollary 2. *There exists minimal surfaces of Catenoid type embedded outside a compact set of* $\mathbb{R}^3$.

Proof. Fix $A, B \in \mathbb{C} - \{0\}$, $AB \in \mathbb{R}$, and use Lemma 1 to find $\lambda, \mu \in \mathbb{C} - \{0\}$ such that the polynomial functions P, Q given by (10) satisfy (11).

Define now $M = \mathbb{C} - \{0, a_1, \dots, a_k, b_1, \dots, b_k\}$ and g, ω as in (5).

The third coordinate function of the minimal surface constructed via the Weierstrass representation is equal to $AB \log|z|$. The result follows from (11). Q.E.D.

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