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# ON INDECOMPOSABLE MODULES OVER DIRECTED ALGEBRAS

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ABSTRACT. Generalizing a result of Bongartz we show that any nonsimple indecomposable module over a finite-dimensional $k$-algebra $A$ is an extension of an indecomposable and a simple module provided $k$ is a field with more than two elements and $A$ is representation directed. Our proof is based on fibre sums over simple modules and some known classification results on socle projective modules over peak algebras. In case the global dimension of $A$ is at most 2 our methods also yield a description of the dimension vectors of the indecomposable $A$-modules by the roots of the associated quadratic form.

## 1. INTRODUCTION AND RESULTS

In [Bo2] Bongartz showed that if $k$ is an algebraically closed field and $A$ is a finite-dimensional $k$-algebra of finite representation type then any finitely-generated nonsimple indecomposable $A$-module $U$ is an extension of an indecomposable and a simple $A$-module. The proof of this result is done by reducing the problem to the representation directed case (cf. [R1]) with the use of covering theory (cf. [BG]) and then solving it by geometric arguments. In [R2] the result recently was extended to representation finite hereditary algebras over finite fields with more than two elements. On the other hand it is known that the directed hereditary quiver algebra $kQ$, where $Q$ is the tree of Dynkin type $D_4$ with all arrows ending in one point has an indecomposable module of dimension 5 which is a counterexample if $k$ is the field with two elements [DR2].

In order to establish the result for arbitrary representation directed algebras over a field $k$ with more than two elements we use fibre sums over simple modules introduced in [M] and further developed in [D2, D3] to reduce the question to categories of socle projective modules over left peak $k$-algebras. Here we can use the classification results given in [K, KS]. Putting things together we obtain in fact the existence of certain exact sequences which as we will see later have some more applications. We want to present the sequences in Proposition 1.1. Before introducing some notations, we would like to express

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We assume that the given representation directed $k$-algebra $A$ is basic and choose a decomposition $1 = \sum_{i=1}^{r} e_i$ of the unit element of $A$ into primitive orthogonal idempotents. We denote the category of finitely-generated left $A$-modules by $A$-mod and recall that the modules $S_i := Ae_i/\operatorname{Rad} Ae_i$ form a complete set of representatives of the isomorphism classes of simple objects in $A$-mod. For a given module $M \in A$-mod the vector $\dim M \in \mathbb{Z}^r$ counting for each $i = 1, \dots, r$ the number of factors isomorphic to $S_i$ occurring in a composition series of $M$ is usually called dimension vector of $M$. As $A$ is directed we know that $\operatorname{gl\,dim} A < \infty$ and hence we have the bilinear map $\langle -, - \rangle : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$ defined by $\langle \dim M, \dim N \rangle := \sum_{j=0}^{\infty} (-1)^j \dim_k \operatorname{Ext}_A^j(M, N)$ where $M, N \in A$-mod. If we denote by $\varepsilon_i = \dim S_i$ the $i$th canonical base vector of $\mathbb{Z}^r$ and consider also the symmetrization $(-, -)$ of the usually nonsymmetric bilinear map $\langle -, - \rangle$ then the reflection at the hyperplane orthogonal to $\varepsilon_i$ with respect to $(-, -)$ is defined by $\sigma_i : \mathbb{Z}^r \to \mathbb{Z}^r$, $x \mapsto x - 2(x, \varepsilon_i)\varepsilon_i/(\varepsilon_i, \varepsilon_i)$.

**Proposition 1.1.** *Suppose $A$ is a representation directed algebra of global dimension at most 2 over a field $k$ with more than two elements. Let $U \in A$-mod be an indecomposable module with dimension vector $d = (d_i)_{i=1,\dots,r}$. Then for each $i = 1, \dots, r$ we have the following result concerning the value $s := 2(d, \varepsilon_i)/(\varepsilon_i, \varepsilon_i)$:*

(a) *If $U$ is not simple, $d_i \neq 0$, and $s > 0$ then*

(a1) *there exists an exact sequence $0 \to U_1 \to U \to U_3 \to 0$ in $A$-mod such that either $U_1 = S_i$ and $U_3$ is indecomposable or $U_1$ is indecomposable and $U_3 = S_i$;*

(a2) *there is an exact sequence $0 \to U_1 \to U \to U_3 \to 0$ in $A$-mod such that either $U_1 = S_i^s$ and $U_3$ is indecomposable or $U_1$ is indecomposable and $U_3 = S_i^s$.*

(b) *If $s < 0$ then there exists an exact sequence $0 \to U_1 \to U_2 \to U_3 \to 0$ such that $U_2$ is indecomposable and either $U_1 = S_i^{-s}$ and $U_3 = U$ or $U_1 = U$ and $U_3 = S_i^{-s}$.*

The proof of this proposition will be given in §2 of this paper. The announced generalization of the result of Bongartz on directed algebras is now easy to derive.

**Theorem 1.2.** *Let $A$ be a representation directed finite-dimensional algebra $A$ over a field $k$ with more than two elements. If $U \in A$-mod is indecomposable and nonsimple then there is an exact sequence $0 \to U_1 \to U \to U_3 \to 0$ in $A$-mod such that $U_1$ and $U_3$ are indecomposable and $U_1$ or $U_3$ is simple.*

*Proof.* By passing to the support algebra of $U$ we may assume that $A$ is sincere (i.e. $e_i U \neq 0$ for all $i = 1, \dots, r$) and hence $\operatorname{gl\,dim} A \leq 2$. Setting $d := \dim U$ we borrow from [Bo2] the inequality $0 < \dim_k \operatorname{End}_A(U) = (d, d) = \sum_{i=1}^{r} (d, \varepsilon_i) d_i$ and find the existence of an index $i$ with $(d, \varepsilon_i) > 0$ and $d_i > 0$. The assertion follows directly from 1.1(a1).

We used only part (a1) of our proposition for this proof. The other two parts are needed to show the next theorem which is well known in case $k$ is algebraically closed (cf. [Bo1, R1]) but without restriction on the field $k$ seems only to be known for hereditary $A$ (cf. [DR1]). To formulate our result we set $R_0 := \{\varepsilon_1, \dots, \varepsilon_r\}$ and define $R_n$ for all $n \in \mathbb{N}$ as the set of all nonzero vectors $x \in \mathbb{Z}^r$ such that $x$ has no negative entries and there exists an $i \in \{1, \dots, r\}$ with $\sigma_i^{-1}(x) \in R_{n-1}$. We call the union $R$ of all sets $R_n$ the set of positive roots of the quadratic form $q_A$ associated to $(-, -)$.

**Theorem 1.3.** *Let $A$ be a representation directed algebra of global dimension at most 2 over a field $k$ with more than two elements. Then the map* dim *is a bijection from the set of all isomorphism classes of indecomposable modules $U \in A$-*mod *onto the set of positive roots of $q_A$.*

*Proof.* As the injectivity of dim is clear by [H] we only have to show that the set of dimension vectors of indecomposables coincides with $R$.

We first prove that each $\dim U$ with $U$ indecomposable is a root and use induction on the length $l(U)$ of $U$. If $l(U) = 1$ then $U = S_i$ for some $i$ and $\dim U = \varepsilon_i \in R_0$.

If $l(U) > 1$ we set $d := \dim U$ and derive from $0 < (d, d)$ as in the proof of 1.2 that there has to be an $i$ with $(d, \varepsilon_i) > 0$ and $d_i > 0$. Using the notations of 1.1 we have $s > 0$ and we find an extension $0 \to U_1 \to U \to U_3 \to 0$ where without loss of generality $U_1 = S_i^s$ and $U_3$ is indecomposable. As $l(U_3) < l(U)$ we know by induction that $d' := \dim U_3 \in R_{n-1}$ for some $n$. From the equation $d = d' + s\varepsilon_i$ now follows $(d', \varepsilon_i) = -(d, \varepsilon_i)$ and this means $d = \sigma_i(d') \in R_n$.

To prove that each root is the dimension vector of an indecomposable module we use induction on $n$. For $n = 0$ again nothing needs to be proven. If $d' \in R_n$ with $n > 0$ we have by definition a vector $d \in R_{n-1}$ such that $\sigma_i(d) = d'$ and by induction an indecomposable module $U$ with $\dim U = d$.

We now distinguish two cases. If $s = 2(d, \varepsilon_i)/(\varepsilon_i, \varepsilon_i) < 0$ using 1.1(b) we can assume that we find an exact sequence $0 \to S_i^{-s} \to U_2 \to U \to 0$ with $U_2$ indecomposable. Hence $\dim U_2 = d - s\varepsilon_i = d'$.

In the other case $s > 0$ we first notice that $0 \le d_i' = d_i - s$. Therefore $d_i > 0$ and $U$ is not simple. So we can apply 1.1(a2) and obtain without loss of generality an exact sequence $0 \to S_i^s \to U \to U_3 \to 0$ with $U_3$ indecomposable. It follows that $\dim U_3 = d - s\varepsilon_i = d'$.

## 2. Proof of Proposition 1.1

Let us first remark that as we are considering left modules we write homomorphisms on the right side of the arguments and therefore, compose them from left to right. We consider a simple module $S \in A$-mod and in order to apply the fibre sum functor with respect to $S$ introduced in [D2] we choose a set $\{V_1, \dots, V_m\}$ of representatives of the isomorphism classes of indecompos-

able modules $V$ such that $\mathrm{Hom}_A(S, V) \neq 0$ but $\mathrm{Ext}^1_A(V, S) = 0$. We assume $V_1 = S$ and denote by $B$ the factor algebra of $\mathrm{End}_A(\bigoplus_{i=1}^m V_i)$ with respect to the ideal of all endomorphisms $f$ with $\mathrm{Hom}_A(S, f) = 0$. If we call $\omega$ the idempotent of $B$ induced by the projection from $\bigoplus_{i=1}^m V_i$ onto the summand $V_1$ and set $T := B\omega$ then we observe that the skew field $F := \mathrm{End}_A(S)$ equals $\mathrm{End}_B(T) = \omega B \omega$. Furthermore $B$ is a schurian left peak $k$-algebra with peak idempotent $\omega$, that is, $T$ is simple projective, $\mathrm{Soc}({}_B B) = \omega\,\mathrm{Soc}({}_B B)$ and for each primitive idempotent $e$ of $B$ the algebra $eBe$ is a skew field. Denoting by $B\text{-}\mathrm{mod}_\omega$ the full subcategory of $B\text{-}\mathrm{mod}$ induced by all modules with projective socle, we summarize in the following remark the results of [D2 and D3] which we will need in this paper.

*Remark* 2.1. There is an equivalence $H$ from $B\text{-}\mathrm{mod}_\omega$ onto the factor category of $A\text{-}\mathrm{mod}$ with respect to the ideal of all homomorphisms $f$ such that $\mathrm{Hom}_A(S, f) = 0$. $H$ maps $T$ to $S$ and has the following properties:

(a) For each indecomposable module $X \in B\text{-}\mathrm{mod}_\omega$ we have that $H(X) \in A\text{-}\mathrm{mod}$ is indecomposable, $\dim_F \mathrm{Hom}_B(T, X) = \dim_F \mathrm{Hom}_A(S, H(X))$ and $H\tau(X) \cong \tau H(X)$ where $\tau$ denotes the Auslander-Reiten translate in the respective categories.

(b) Let $0 \to K \xrightarrow{f} X \xrightarrow{g} Y \to 0$ be an exact sequence in $B\text{-}\mathrm{mod}_\omega$ with $T$ not direct summand of $X$ and $Y$. If $f(K) \subset \mathrm{Rad}\, X$ and $\mathrm{Soc}(K) \cong T^q$ for some $q \in \mathbb{N}$ then we obtain an exact sequence $0 \to S^q \to H(X) \xrightarrow{H(g)} H(Y) \to 0$ in $A\text{-}\mathrm{mod}$.

The existence of $H$ is proven in [D3, Folgerung] whereas the properties of $H$ follow from the construction of $H$ and [D2, 3.5]. In the next lemma we look for certain exact sequences in $B\text{-}\mathrm{mod}_\omega$.

**Lemma 2.2.** *Suppose $B$ is a schurian sp-representation finite left peak algebra with peak idempotent $\omega$ over a field $k$ with more than two elements. Let $X \in B\text{-}\mathrm{mod}_\omega$ be indecomposable and not isomorphic to $T = B\omega$. Using the notations $F := \mathrm{End}_B(T)$ and $\Delta := \dim_F \mathrm{Hom}_B(T, X) - \dim_F \mathrm{Hom}_B(T, \tau X)$ we have the following results*:

(a1) *If either* (i) $\Delta > 1$ *or* (ii) $\Delta = 1$ *and $X$ is not projective then there is an exact sequence $0 \to K \xrightarrow{f} X \xrightarrow{g} Y \to 0$ in $B\text{-}\mathrm{mod}_\omega$ with the properties that $Y$ is indecomposable, $T \ncong Y$, $f(K) \subset \mathrm{Rad}\, X$, and $\mathrm{Soc}\, K \cong T$.*

(a2) *If $\Delta > 0$ and $X$ is not projective then there is an exact sequence $0 \to K \xrightarrow{f} X \xrightarrow{g} Y \to 0$ in $B\text{-}\mathrm{mod}_\omega$ with the properties that $Y$ is indecomposable, $T \ncong Y$, $f(K) \subset \mathrm{Rad}\, X$, and $\mathrm{Soc}\, K \cong T^\Delta$.*

(b) *If $\Delta < 0$ then there is an exact sequence $0 \to K \xrightarrow{f} Y \xrightarrow{g} X \to 0$ in $B\text{-}\mathrm{mod}_\omega$ with the properties that $Y$ is indecomposable, $T \ncong Y$, $f(K) \subset \mathrm{Rad}\, Y$, $\mathrm{Soc}\, K \cong T^{-\Delta}$, and $\dim_F \mathrm{Hom}_B(T, Y) - \dim_F \mathrm{Hom}_B(T, \tau Y) = -\Delta$.*

*Proof*. We want to reduce the problem to the sp-sincere case. For this reason let $1 = \sum_{j=1}^{s} u_j$ be a decomposition of the unit element of $B$ into primitive orthogonal idempotents. We assume that $u_s = \omega$ and $u_j(X/\operatorname{Rad} X) \neq 0$ holds exactly for $j = 1, \ldots, t$. After setting $u := \sum_{j=1}^{t} u_j + \omega$ we consider the left peak algebra $B' := uBu$. The canonical restriction functor $R\colon B\text{-}\mathrm{mod}_\omega \to B'\text{-}\mathrm{mod}_\omega$ has the properties that $R(X)$ is indecomposable and $R(X)$ is projective iff $X$ is. Defining $T' := B'\omega$ we have $F = \operatorname{End}_{B'}(T')$ and $\dim_F \operatorname{Hom}_B(T, X) = \dim_F \operatorname{Hom}_{B'}(T', R(X))$. Moreover from [Bu] follows the equality $\dim_F \operatorname{Hom}_B(T, \tau_B X) = \dim_F \operatorname{Hom}_{B'}(T', \tau_{B'} R(X))$.

Our next step is to produce the desired sequences for $R(X)$ in the category $B'\text{-}\mathrm{mod}_\omega$. But by construction $R(X)$ is sp-sincere and hence all possibilities for $B'$ and $R(X)$ are listed in [KS]. Actually by a case by case inspection we find suitable sequences for $R(X)$ where not only $\operatorname{Soc} K$ but $K$ itself is isomorphic to a product of appropriately many copies of $T'$. The problem is especially handy in the homogeneous cases, that is, the exact partially ordered sets (cf. [K]) because finding the exact sequences in these cases can be reformulated to finding solutions of certain linear equations over the field $k$.

We also note that for finite field $k$ with $q \neq 2$ elements the existence of the sequences follows from [R2]. In fact in this paper all possible sequences for given $X$ are counted and the number is just the evaluation of the corresponding Hall polynomial at $q$. The nonexistence of the sequences for $|k| = 2$ is reflected by the fact that 2 is the only prime power which is a root of some Hall polynomial.

All we have to do now to finish the proof of Lemma 2.2 is to apply the left adjoint $L\colon B'\text{-}\mathrm{mod}_\omega \to B\text{-}\mathrm{mod}_\omega$ of $R$ to our sequences. it is easy to see that $LR(X) \cong X$ and the image sequences have the desired properties.

In order to complete the proof of Proposition 1.1 we assume $S = S_i$ and define $f := \dim_k F = (\varepsilon_i, \varepsilon_i)$. We start to prove (a1) and realize that because $d_i \neq 0$ the index $i$ is contained in the support $I := \{j : e_j U \neq 0\}$ of $U$. By [Bol, 3.2] the set $I$ lies convex inside the quiver of $A$. If we consider the idempotent $e := \sum_{j \in I} e_j$ of $A$ and the algebra $A' := eAe$ then $U$ is by construction a sincere $A'$-module and hence of injective and projective dimension at most 1. The convexity of $I$ by [D1, Lemma] therefore yields the equations $\operatorname{Ext}_A^2(S, U) = \operatorname{Ext}_{A'}^2(S, U) = 0$ and $\operatorname{Ext}_A^2(U, S) = \operatorname{Ext}_{A'}^2(U, S) = 0$.

In order to rewrite the $\operatorname{Ext}_A^1$-terms occurring in the expression $2(d, \varepsilon_i)$ we derive from [AR, 3.3] the identities $\dim_k \operatorname{Ext}_A^1(S, U) = \dim_k \operatorname{Hom}_A(\tau^- U, S)$ and $\dim_k \operatorname{Ext}_A^1(U, S) = \dim_k \operatorname{Hom}_A(S, \tau U)$ using that $S$ is simple and therefore, no nontrivial factoring through projective respectively injective modules can occur. Possibly passing to the dual algebra the inequality $0 < 2(d, \varepsilon_i)$ allows us to assume $\operatorname{Hom}_A(S, U) \neq 0$. But as $A$ is directed then immediately follows $\operatorname{Hom}_A(U, S) = 0 = \operatorname{Hom}_A(\tau^- U, S)$. Altogether we obtain $0 < s = \dim_F \operatorname{Hom}_A(S, U) - \dim_F \operatorname{Hom}_A(S, \tau U)$. Using 2.1 we find an indecompos-

able module $X \in B\text{-mod}_\omega$ with the properties $H(X) = U$, $X \ncong T$ because $U$ is not simple and $\Delta := \dim_F \operatorname{Hom}_B(T, X) - \dim_F \operatorname{Hom}_B(T, \tau X) = s > 0$. If $X$ is not projective or $\Delta > 1$ then 2.2(a1) and 2.1(b) show the assertion. If $X$ is projective and $\Delta = 1$ then $\dim_F \operatorname{Hom}_A(S, U) = 1$ and by choosing a monomorphism $h: S \to U$ we find an exact sequence $0 \to S \xrightarrow{h} U \to U' \to 0$ with indecomposable end term $U'$ (cf. [D4, 2.3.e)]).

We skip the proof of (a2) which is similar to (a1). For the proof of (b) we consider the inequality $0 > sf = \dim_k \operatorname{Hom}_A(S, U) + \dim_k \operatorname{Hom}_A(U, S) - \dim_k \operatorname{Hom}_A(\tau^- U, S) - \dim_k \operatorname{Hom}_A(S, \tau U) + \rho$. We again applied the formulas for the $\operatorname{Ext}^1_A$-terms from above and introduced the abbreviation

$$\rho := \dim_k \operatorname{Ext}^2_A(S, U) + \dim_k \operatorname{Ext}^2_A(U, S) \geq 0.$$

Again passing to the dual algebra if necessary we may assume $\operatorname{Hom}_A(S, \tau U) \neq 0$. That $A$ is directed allows us to derive $\operatorname{Hom}_A(U, S) = 0 = \operatorname{Hom}_A(\tau^- U, S)$. Hence we find

$$0 > s = \dim_F \operatorname{Hom}_A(S, U) - \dim_F \operatorname{Hom}_A(S, \tau U) + (\rho/f).$$

We distinguish two cases and start with the case $\operatorname{Hom}_A(S, U) \neq 0$. Remark 2.1 provides us with an indecomposable module $X \in B\text{-mod}_\omega$ satisfying $H(X) = U$. As $U$ is not simple $X$ is not isomorphic to $T$. 2.1(a) shows that $\Delta := \dim_F \operatorname{Hom}_B(T, X) - \dim_F \operatorname{Hom}_B(T, \tau X) < 0$. Using 2.2(b) and 2.1(b) we obtain an exact sequence $0 \to S^{-\Delta} \to U' \to U \to 0$ with $U'$ indecomposable and $-\Delta = \dim_F \operatorname{Hom}_A(S, U') - \dim_F \operatorname{Hom}_A(S, \tau U')$.

As $i$ has to lie in the support of $U'$ we can argue as above to get $\operatorname{Ext}^2_A(S, U') = 0 = \operatorname{Ext}^2_A(U', S)$. Furthermore because $A$ is directed and $\operatorname{Hom}_A(S, U') \neq 0$ we have $\operatorname{Hom}_A(U', S) = 0 = \operatorname{Hom}_A(\tau^- U', S)$ and hence for $d' := \dim U'$ the equality $-\Delta = 2(d', \varepsilon_i)/(\varepsilon_i, \varepsilon_i)$ holds. An easy calculation shows $\Delta = 2(d, \varepsilon_i)/(\varepsilon_i, \varepsilon_i) = s$.

For the second case we have to assume $\operatorname{Hom}_A(S, U) = 0$. From $\operatorname{Ext}^1_A(U, S) \neq 0$ follows the existence of a nonsplit exact sequence $0 \to S \to U' \to U \to 0$ such that each indecomposable summand $U''$ of $U'$ satisfies $\operatorname{Hom}_A(S, U'') \neq 0$. Therefore each $U''$ is contained in the image of $H$ and as in the proof of [D2, Lemma 3.1] we can construct an exact sequence $0 \to S^q \xrightarrow{h} V \to U \to 0$ such that all indecomposable summands $V'$ of $V$ lie in the set $\{V_2, \ldots, V_m\}$. By considering for each of these summands $V'$ the largest submodule $S(V')$ isomorphic to a product of copies of $S$ and defining $\overline{V}' := V'/S(V')$ it is easy to see that $U$ is the direct sum of the indecomposable modules $\overline{V}'$. Hence $V$ itself has to coincide with some $V_j$ and thus we have $q = \dim_F \operatorname{Hom}_A(S, V_j)$. The exact sequence

$$0 = \operatorname{Hom}_A(V_j, S) \to \operatorname{Hom}_A(S^q, S) \to \operatorname{Ext}^1_A(U, S) \to \operatorname{Ext}^1_A(V_j, S) = 0$$

shows that $q = \dim_F \operatorname{Hom}_A(S^q, S) = \dim_F \operatorname{Ext}^1_A(U, S) = -\Delta$. Now we can continue as in the first case.

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