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HARMONIC TWO-FORMS IN FOUR DIMENSIONS

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ABSTRACT. Conformal invariance of middle-dimensional harmonic forms is used to improve Kato's inequality for four-manifolds. An application to positively curved four-manifolds is given.

0. INTRODUCTION

The purpose of this paper is to prove the following:

Theorem 1. *Let (M^4, g) be a four-dimensional Riemannian manifold. Let ω be a harmonic two-form on (M, g). Then ω satisfies the pointwise inequality:*

$$|\nabla\omega|^2 \geq \tfrac{3}{2}|d|\omega||^2. \tag{0.1}$$

Kato's inequality [1, p. 130], states that if E is a Riemannian vector bundle with connection $\overline{\nabla}$ over a Riemannian manifold M, then any smooth section s of E, satisfies the pointwise inequality:

$$|\overline{\nabla s}|^2 \geq |d|s||^2. \tag{0.2}$$

Now by definition, if $s(\omega)$ vanishes at $p \in M$, then $d|s|(d|\omega|) = 0$ at p. Thus, (0.1) and (0.2) are automatically valid at such a point. At points where ω does not vanish (0.1) can be thought of as a quantitative improvement of (0.2), for the case of harmonic two-forms on four-dimensional manifolds.

As an application of the above theorem, we prove:

Theorem 2. *Let (M^4, g) be a compact, connected four-dimensional Riemannian manifold whose sectional curvature $K(g)$ satisfies $1 \geq K(g) \geq \delta$. If*

$$\delta \geq 1/(3(1 + 3\cdot 2^{1/4}/5^{1/2})^{1/2} + 1) \approx .1714 \tag{0.3}$$

then M is definite.

This theorem represents an improvement of results starting with [2] followed by [4, 7, 6]. The relevance of Theorems 1 and 2 stems from the following facts

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(cf. [2, 4, 7, 6]): the "Sphere Theorem" classifies compact simply-connected Riemannian manifolds whose sectional curvature K satisfies $1 \geq K \geq \frac{1}{4}$, while the Hopf conjecture states that $\mathbb{S}^2 \times \mathbb{S}^2$ does not admit a strictly positively curved metric. In Theorem 2, the curvature pinching is below $\frac{1}{4}$, and the resulting M must be definite. As $\mathbb{S}^2 \times \mathbb{S}^2$, as well as $\mathbb{S}^2 \times \mathbb{S}^2 \sharp \mathbb{C}P^2 \sharp \cdots \sharp \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} \sharp \cdots \sharp \overline{\mathbb{C}P^2}$ (connected sum, where $\overline{\mathbb{C}P^2}$ means $\mathbb{C}P^2$ with the usual orientation reversed) are all indefinite, these manifolds thus cannot admit metrics with sectional curvature so pinched as to satisfy 0.3. At this point, we are not able to prove that an M satisfying the above hypotheses is topologically $\mathbb{S}^4$ or $\mathbb{C}P^2$, since the arguments in [7] do not appear to extend down to the pinching in Theorem 2.

The idea behind Theorem 1 is as follows: let M_0 be the open subset of M where ω does not vanish. Change g conformally to a new metric g' on M_0 relative to which ω has constant length. By the conformal invariance of middle-dimensional harmonic forms, ω is still harmonic for the Riemannian manifold (M_0, g'). Writing out the formula ((1.8) §1) for R_2', the Weitzenbock operator on two-forms for g', in terms of R_2, the corresponding operator for g, and using the harmonicity of ω for either metric, proves (0.1) on M_0. As mentioned, this then proves (0.1) on all of M.

1. Background and proofs

Because there are many notational and sign differences in use for various geometric objects, we will first establish the notation to be used in this paper.

If (M, g) is a Riemannian manifold, with Riemannian connection ∇, then the curvature tensor, R considered as a $(3, 1)$ tensor, is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \tag{1.1}$$

where X, Y, Z are vector fields on M. For a plane P spanned by (orthonormal) $X_p, Y_p \in T_pM$, the sectional curvature of P is given by

$$K(P) = g_p(R(X, Y)Y, X) = \langle R(X, Y)Y, X \rangle_p. \tag{1.2}$$

Aside from (1.1) and (1.2), our conventions are the same as in [3].

If f is a smooth function on M, and g' is the metric given by $g' = e^{2f} g$, then, considered as a $(4, 0)$ tensor, we have, for R', the curvature tensor of g':

$$R' = e^{2f}(R + g \owedge (D\,df - df \circ df + \tfrac{1}{2}|df|^2 g)) \tag{1.3}$$

(cf. [3, p. 58], the sign difference comes from (1.1), but our Ricci tensors are the same). The Weitzenbock operator R_2 of (M, g), is the $(4, 0)$-tensor given by (cf. [5])

$$R_2 = \mathrm{Ric} \owedge g + 2R. \tag{1.4}$$

That is, at a point $p \in M$, with $v_i, w_i \in T_pM$, $i = 1, 2$, we define

(1.4.1)
$$\begin{aligned} R_2(v_1, v_2, w_1, w_2) = {} & \mathrm{Ric}(v_1, w_1)g(v_2, w_2) + \mathrm{Ric}(v_2, w_2)g(v_1, w_1) \\ & - \mathrm{Ric}(v_1, w_2)g(v_2, w_1) - \mathrm{Ric}(v_2, w_1)g(v_1, w_2) \\ & + 2\langle R(v_1, v_2)w_1, w_2\rangle . \end{aligned}$$

We also take (1.4.1) as the definition of R_2 considered as a map from two-vectors to two-vectors:

$$\langle R_2(v_1 \wedge v_2), w_1 \wedge w_2\rangle = R_2(v_1, v_2, w_1, w_2). \tag{1.4.2}$$

Finally, R_2 is considered as a map from two-forms to two-forms by

$$R_2(\omega)(v_1 \wedge v_2) = \omega(R_2(v_1 \wedge v_2)), \tag{1.4.3}$$

where ω is a two-form at p. Considering R_2 as a map from two-forms to two-forms as in (1.4.3), one has (cf. [6]).

$$\langle \Delta\omega, \omega\rangle = \tfrac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 + \langle R_2\omega, \omega\rangle, \tag{1.5}$$

where $|\nabla\omega|^2$ is the length of the tensor $\nabla\omega$, and $\langle\ ,\ \rangle$ denotes the inner product induced on two-*forms* from g. For the remainder of the paper, we assume M is four dimensional. In this case, if $\langle\ ,\ \rangle'$ is the inner product on two-forms from $g' = e^{2f}g$, then

$$\langle\ ,\ \rangle' = e^{-4f}\langle\ ,\ \rangle \tag{1.6}$$

whereas for the induced inner product on two-vectors, one has

$$\langle\ ,\ \rangle' = e^{4f}\langle\ ,\ \rangle. \tag{1.6.1}$$

Using (1.3) and (d), p. 59, [3], with $n = 4$, we conclude:

$$R_2' = e^{2f}[R_2 + (\Delta f - |df|^2)g \owedge g] \quad \text{as } (4, 0)\text{-tensors}. \tag{1.7}$$

For R_2 considered as a map of two-vectors, one has

$$R_2' = e^{-2f}(R_2 + 2(\Delta f - |df|^2)\mathrm{Id}); \tag{1.7.1}$$

and for R_2 considered as a map of two-forms the relation between R_2 and R_2' is the same as (1.7.1). Using (1.7.1) and (1.6), one finds

$$\langle R_2'\omega, \omega\rangle' = e^{-6f}\{\langle R_2\omega, \omega\rangle + 2(\Delta f - |df|^2)|\omega|^2\}, \tag{1.8}$$

where ω is a two-form.

Fix any two-form on M, and let M_0 be the open subset $M_0 = \{p \in M | \omega_p \neq 0\}$. On M_0, $|\omega|$ is a smooth function. Endow M_0 with the Riemannian metric

$$g' = |\omega|g, \quad (\text{i.e., } e^{2f} = |\omega|) \tag{1.9}$$

so $\langle\omega, \omega\rangle' \equiv 1$. In this case, one computes that

$$2(\Delta f - |df|^2) = \frac{\Delta|\omega|}{|\omega|} + \frac{1}{2}\frac{|d|\omega||^2}{|\omega|^2} \quad (\text{on } M_0)$$

so, using $\frac{1}{2}\Delta|\omega|^2 = |\omega|\Delta|\omega| - |d|\omega||^2$, we see that (1.8) yields

$$\langle R_2', \omega, \omega\rangle' = |\omega|^{-3}\{\langle R_2\omega, \omega\rangle + \tfrac{1}{2}\Delta|\omega|^2 + \tfrac{3}{2}|d|\omega||^2\}. \tag{1.10}$$

Proof of Theorem 1. Let ω be a harmonic two-form on (M, g). Then ω is also harmonic on (M_0, g) and, since middle-dimensional harmonic forms are conformally invariant, ω is harmonic on (M_0, g') with g' given by (1.9). Since ω is g harmonic, (1.5) yields $\langle R_2\omega, \omega\rangle + \frac{1}{2}\Delta|\omega|^2 = -|\nabla\omega|^2$. Since ω is g' harmonic, with constant length, (1.5) also yields $\langle R_2'\omega, \omega\rangle' = -|\nabla'\omega|'^2$. Thus (1.10) yields

$$-|\nabla'\omega|'^2 = |\omega|^{-3}\{-|\nabla\omega|^2 + \tfrac{3}{2}|d|\omega||^2\} \tag{1.11}$$

which proves (0.1) on M_0, and thus on M. Q.E.D.

The proof of Theorem 2 follows the same line of argument as [6]. One inserts (0.1) in the arguments of [6] between (4) and (5) to conclude (in the notation of that paper):

$$|\nabla(|X_+| - |X_-|)|^2 \le \tfrac{4}{3}|\nabla X|^2. \tag{1.12}$$

Here X is a two-form on a four-manifold with X_+, X_- the selfdual and anti-self-dual components, respectively. Substituting (1.12) into (5) of [6] yields

$$\int \frac{4}{3}|\nabla X|^2 \ge \lambda_1 \int (|X_+| - |X_-|)^2 \tag{1.13}$$

and the remainder of the proof proceeds as in that paper.

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