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ON MEASURABLE LOCAL HOMOMORPHISMS

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ABSTRACT. We prove that every measurable local homomorphism between locally compact groups is continuous.

Let G and H be locally compact groups. In a recent paper A. Kleppner proved that every measurable homomorphism from G into H is continuous [2]. In the present note we show that this result is true for local homomorphisms as well. Our method of proof differs from that of [2]. The measure referred to throughout is a left Haar measure on G.

Theorem. *Let G and H be locally compact groups and let V be an open neighborhood of the identity of G. Suppose that φ is a mapping from V into H such that*

(i) $\varphi(xy) = \varphi(x)\varphi(y)$ *whenever* $x, y, xy \in V$;
(ii) $\varphi^{-1}(U)$ *is measurable for every open set* $U \subset H$.

Then φ is continuous.

Proof. Let U be an arbitrary neighborhood of the identity of H. To show φ is continuous it is sufficient to show that $\varphi^{-1}(U)$ contains a neighborhood of the identity of G. Choose a symmetric, relatively compact open set $W \subset U$, $W \neq \varnothing$ so that $WW \subset U$. If $\varphi^{-1}(W)$ is not locally null then by Corollary (20.17) in [1] the set $\varphi^{-1}(W)\varphi^{-1}(W)^{-1}$ contains a neighborhood of the identity. Since

$$\varphi^{-1}(W)\varphi^{-1}(W)^{-1} \subset \varphi^{-1}(WW) \subset \varphi^{-1}(U),$$

it remains to prove that $\varphi^{-1}(W)$ cannot be locally null.

On the other hand, suppose that $\varphi^{-1}(W)$ is locally null, and denote by H_0 the open subgroup generated by W. Choose a symmetric, relatively compact open set $V_0 \subset V$, $V_0 \neq \varnothing$ with $V_0V_0 \subset V$. We show that $V_0 \cap \varphi^{-1}(xH_0)$ is a null set for every $x \in H$.

Since xH_0 is σ-compact it can be covered by denumerably many sets of the form yW $(y \in H)$. Thus, it suffices to prove that $V_0 \cap \varphi^{-1}(yW)$ is a null set for every $y \in H$. We put $E := \varphi(V_0)$ and $S := (E \cap yW)^{-}$. Because S is compact

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and $S \subset \bigcup_{z\in E} zW$, there exists a finite number of elements $z_1, \dots, z_n \in E$ such that

$$S \subset \bigcup_{k=1}^{n} z_k W.$$

We have

$$\begin{aligned} V_0 \cap \varphi^{-1}(yW) &\subset V_0 \cap \varphi^{-1}(S) \subset V_0 \cap \left(\bigcup_{k=1}^{n} \varphi^{-1}(z_k W)\right) \\ &= V_0 \cap \left(\bigcup_{k=1}^{n} \varphi^{-1}(E \cap z_k W)\right). \end{aligned} \tag{1}$$

Choose $g_1, \dots, g_n \in V_0$ so that $\varphi(g_k) = z_k$ $(k = 1, \dots, n)$. Using the relation $V_0^{-1}V_0 = V_0V_0 \subset V$ it is not difficult to see that

$$V_0 \cap \varphi^{-1}(E \cap z_k W) = V_0 \cap g_k\varphi^{-1}(W). \tag{2}$$

It follows immediately from (1) and (2) that $V_0 \cap \varphi^{-1}(yW)$ is a null set.

Now let x_α $(\alpha \in \Gamma)$ be the set of left coset representatives of H_0. For an arbitrary index set $\Gamma_0 \subset \Gamma$, the set $\bigcup_{\alpha\in\Gamma_0} x_\alpha H_0$ is open; so, $\bigcup_{\alpha\in\Gamma_0}(V_0 \cap \varphi^{-1}(x_\alpha H_0))$ is measurable. Moreover,

$$V_0 = \bigcup_{\alpha\in\Gamma}(V_0 \cap \varphi^{-1}(x_\alpha H_0)),$$

where the sets $V_0 \cap \varphi^{-1}(x_\alpha H_0)$ are pairwise disjoint null sets. It follows from Theorem 3.1 in [3] that V_0 is a null set. This contradiction shows that $\varphi^{-1}(W)$ cannot be locally null.

References

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