ON MEASURABLE LOCAL HOMOMORPHISMS

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Abstract. We prove that every measurable local homomorphism between locally compact groups is continuous.

Let $G$ and $H$ be locally compact groups. In a recent paper A. Kleppner proved that every measurable homomorphism from $G$ into $H$ is continuous [2]. In the present note we show that this result is true for local homomorphisms as well. Our method of proof differs from that of [2]. The measure referred to throughout is a left Haar measure on $G$.

Theorem. Let $G$ and $H$ be locally compact groups and let $V$ be an open neighborhood of the identity of $G$. Suppose that $\phi$ is a mapping from $V$ into $H$ such that

(i) $\phi(xy) = \phi(x)\phi(y)$ whenever $x, y, xy \in V$;

(ii) $\phi^{-1}(U)$ is measurable for every open set $U \subset H$.

Then $\phi$ is continuous.

Proof. Let $U$ be an arbitrary neighborhood of the identity of $H$. To show $\phi$ is continuous it is sufficient to show that $\phi^{-1}(U)$ contains a neighborhood of the identity of $G$. Choose a symmetric, relatively compact open set $W \subset U$, $W \neq \emptyset$ so that $WW \subset U$. If $\phi^{-1}(W)$ is not locally null then by Corollary (20.17) in [1] the set $\phi^{-1}(W)\phi^{-1}(W)^{-1}$ contains a neighborhood of the identity. Since $\phi^{-1}(W)\phi^{-1}(W)^{-1} \subset \phi^{-1}(WW) \subset \phi^{-1}(U)$,

it remains to prove that $\phi^{-1}(W)$ cannot be locally null.

On the other hand, suppose that $\phi^{-1}(W)$ is locally null, and denote by $H_0$ the open subgroup generated by $W$. Choose a symmetric, relatively compact open set $V_0 \subset V$, $V_0 \neq \emptyset$ with $V_0V_0 \subset V$. We show that $V_0 \cap \phi^{-1}(xH_0)$ is a null set for every $x \in H$.

Since $xH_0$ is $\sigma$-compact it can be covered by denumerably many sets of the form $yW$ ($y \in H$). Thus, it suffices to prove that $V_0 \cap \phi^{-1}(yW)$ is a null set for every $y \in H$. We put $E := \phi(V_0)$ and $S := (E \cap yW)^{-1}$. Because $S$ is compact...
and \( S \subset \bigcup_{z \in E} zW \), there exists a finite number of elements \( z_1, \ldots, z_n \in E \)

such that

\[
S \subset \bigcup_{k=1}^{n} z_k W.
\]

We have

\[
V_0 \cap \varphi^{-1}(yW) \subset V_0 \cap \varphi^{-1}(S) \subset V_0 \cap \left( \bigcup_{k=1}^{n} \varphi^{-1}(z_k W) \right)
\]

\[
= V_0 \cap \left( \bigcup_{k=1}^{n} \varphi^{-1}(E \cap z_k W) \right).  
\]  

Choose \( g_1, \ldots, g_n \in V_0 \) so that \( \varphi(g_k) = z_k \) \((k = 1, \ldots, n)\). Using the relation \( V_0^{-1}V_0 = V_0V_0 \subset V \) it is not difficult to see that

\[
V_0 \cap \varphi^{-1}(E \cap z_k W) = V_0 \cap g_k \varphi^{-1}(W).
\]

It follows immediately from (1) and (2) that \( V_0 \cap \varphi^{-1}(yW) \) is a null set.

Now let \( x_\alpha (\alpha \in \Gamma) \) be the set of left coset representatives of \( H_0 \). For an arbitrary index set \( \Gamma_0 \subset \Gamma \), the set \( \bigcup_{\alpha \in \Gamma_0} x_\alpha H_0 \) is open; so, \( \bigcup_{\alpha \in \Gamma_0} (V_0 \cap \varphi^{-1}(x_\alpha H_0)) \) is measurable. Moreover,

\[
V_0 = \bigcup_{\alpha \in \Gamma} (V_0 \cap \varphi^{-1}(x_\alpha H_0)),
\]

where the sets \( V_0 \cap \varphi^{-1}(x_\alpha H_0) \) are pairwise disjoint null sets. It follows from Theorem 3.1 in [3] that \( V_0 \) is a null set. This contradiction shows that \( \varphi^{-1}(W) \) cannot be locally null.

**References**


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