PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 2, June 1991

HYPERSURFACES IN $\mathbf{R}^n$ WHOSE UNIT NORMAL HAS SMALL BMO NORM

STEPHEN SEMMES

(Communicated by J. Marshall Ash)

ABSTRACT. Let M be a hypersurface in $\mathbf{R}^{d+1}$ whose Gauss map has small BMO norm. This condition is closely related to (but much weaker than) the requirement that the principal curvatures of M have small $L^d(M)$ norm. (The relationship between these two conditions is a nonlinear geometrical analogue of a classical Sobolev embedding.) This paper deals with the problem of understanding the geometrical constraints imposed on M by the requirement that the Gauss map have small BMO norm.

Let M be a connected codimension 1 submanifold of $\mathbf{R}^{d+1}$, and let $n(x)$ denote a choice of unit normal on M. What can we say about M if n has small BMO norm, i.e., if

$$(0.1) \qquad ||n||_* = \sup_{\substack{x\in M\\ R>0}} \frac{1}{|B(x,R)\cap M|}\int_{B(x,R)\cap M} |n(y)-n_{x,R}|\,dy$$

is small? Here $B(x,R)$ denotes the ball with center x and radius R, $|E|$ denotes the surface measure of E if $E\subseteq M$, dy denotes surface measure on M, and

$$n_{x,R} = \frac{1}{|B(x,R)\cap M|}\int_{B(x,R)\cap M} n(y)\,dy.$$

I shall assume a priori that M is smooth, and even that $M\cup\{\infty\}$ is smooth inside of $\mathbf{R}^{d+1}\cup\{\infty\}\cong S^{d+1}$. Of course this assumption will not be used in any quantitative way. Algebraic topology implies that M is orientable and has exactly two complementary components.

Notice that n always lies in BMO, because it takes values in a compact set. If there were some vector v_0 so that $||n-v_0||_{L^\infty(M)}$ were small, then M would have to be a Lipschitz graph (with small constant). However, it is easy to construct examples of M's that spiral around as much as you want—and are thus far from being graphs—but so that $||n||_*$ is as small as you want.

Let me describe two ways in which the condition $||n||_*$ small can arise.

Received by the editors July 10, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42B99; Secondary 53A05.
The author is partially supported by the NSF and the Sloan Foundation.

The first comes from [Se1, 2]. Let us call M a chord-arc surface with small constant if $\|n\|_*$ is small and also

$$\sup_{\substack{x\in M\\ R>0}} \sup_{y\in B(x,R)\cap M} |\langle x-y, n_{x,R}\rangle| R^{-1} \tag{0.2}$$

is small. (In other words, this last holds if for all $x \in M$ and $R > 0$, $B(x, R)\cap M$ stays close to the hyperplane through x normal to $n_{x,R}$.) When $d = 1$, this definition reduces to the well-studied class of chord-arc curves with small constant, which is where the name comes from. (See, e.g., the survey paper [Se3].)

In [Se1] the class of chord-arc surfaces is given various function-theoretic, operator theoretic, and geometrical characterizations. One such characterization requires that the Cauchy integral operator T on M (defined in terms of Clifford analysis) be almost selfadjoint, i.e., $T - T^*$ should have small operator norm on $L^2(M)$. Another characterization requires that

$$\sup_{\substack{x\in M\\ R>0}} \left|\frac{|M\cap B(x,R)|}{\nu^d R^d} - 1\right| + \sup_{x,y\in M}\left|\frac{d(x,y)}{|x-y|} - 1\right|$$

be small, where ν_d denotes the volume of the unit ball in $\mathbf{R}^d$, and $d(x, y)$ denotes the length of the shortest curve on M that joins x to y.

The question naturally arises as to whether (0.2) is automatically small if $\|n\|_*$ is small. When $d = 1$ this is true and easy.

Theorem. *If M is as above, then* (0.2) *is small if $\|n\|_*$ is, and in that case* $(0.2) \leq C\|n\|_*$.

It is an interesting open problem to find some natural parameterization for the space of chord-arc surfaces with small constant in terms of n. When $d = 1$ this is possible and very useful.

Another open problem is whether chord-arc surfaces with small constant can be parameterized particularly nicely by $\mathbf{R}^d$, e.g., if there will always be a bi-Lipschitz parameterization. In [Se2] partial results are obtained; in particular, it is shown that M must be homeomorphic to $\mathbf{R}^d$ if M is a chord-arc surface with small constant and that one can build parameterizations with control on the L^p norm of the gradient, with $p \to \infty$ as $\|n\|_*$ and (0.2) tends to 0.

Let me describe the second context in which the condition $\|n\|_*$ small arises naturally. The differential of n on M can be viewed as a function S on M so that $S(x)$ is a linear transformation on the tangent space to M at x, called the shape operator. It is always symmetric, and its eigenvalues are called the principal curvatures of M at x.

Let $k(x)$ denote the maximum of the principal curvatures of M at x. If $k \in L^d(M)$, then it is natural to expect that $n \in \mathrm{VMO}(M)$ (VMO = vanishing mean oscillation), i.e., that

$$\frac{1}{|B(x,R)\cap M|}\int_{B(x,R)\cap M} |n(y) - n_{x,R}|\, dy \tag{0.3}$$

tends to 0 uniformly as $R \to \infty$, $R \to 0$, or as $x \in M$ tends to ∞. The reason for expecting this is the well-known fact that $f \in \mathrm{VMO}(\mathbf{R}^d)$ if $\nabla f \in L^d(\mathbf{R}^d)$, as one can easily prove using the Poincaré inequality. Thus to show that $n \in \mathrm{VMO}(M)$ if $k \in L^d(M)$, one needs to get a handle on the Poincaré inequality on M. Such estimates are known and are given in terms of the mean curvature of M. See [Si]. In our case—where we assume L^d control on all the principal curvatures—we could also get the appropriate Poincaré inequality using the techniques of this paper. (We shall in fact see that Poincaré and Sobolev estimates hold on any chord-arc surface with small constant.)

Notice that $k \in L^d(M)$ is the critical case: if $k \in L^p$ for $p > d$, then M is locally a $C^{1,\alpha}$ graph for some $\alpha > 0$, with estimates that depend only on p and $||k||_p$ and not on our a priori assumptions. For $p = d$, this breaks down; one can again build M's with $k \in L^d$ which spiral around, or which have arbitrarily small handles. (However, there cannot be too many handles.) Also, $||k||_{L^d(M)}$ is scale invariant—it does not change if we dilate M. This is not the case for $||k||_p$ when $p > d$.

The plan for the proof of the theorem is the following. Set

$$\theta(R) = \sup_{0<r\leq R}\ \sup_{x\in M}\ \sup_{y\in M\cap B(x,r)} |\langle x-y, n_{x,r}\rangle| r^{-1},$$

so that $\theta(\infty)$ equals (0.2). We shall prove that if $||n||_*$ and $\theta(R)$ are sufficiently small (for some fixed R), then

$$\theta(\tfrac{1}{400}R) \leq C||n||_*, \tag{0.4}$$

and that if $||n||_*$ and $\theta(R)$ are again sufficiently small (for some other R), then

$$\theta(\tfrac{3}{2}R) \leq C(\theta(R) + ||n||_*). \tag{0.5}$$

Here "sufficiently small" is not allowed to depend on M or R. It is then easy to show that $\theta(\infty) \leq C||n||_*$ using (0.4), (0.5), and the fact that $\theta(R) \to 0$ as $R \to 0$ (because of our a priori assumptions).

In proving (0.4) and (0.5) we shall use heavily the results of [Se1] (reviewed in §1), which tell us a lot about the geometry of M on the scale of R when $||n||_*$ and $\theta(R)$ are small enough. In particular, the idea for proving (0.4) is as follows. Fix $x \in M$, and set $f(y) = \langle y, n_{x,R}\rangle$ on $M \cap B(x, R)$. Clearly $|\nabla_T f(y)| \leq |n(y) - n_{x,R}|$, where $|\nabla_T f(y)|$ denotes the tangential gradient of f on M at y. Thus we can get control on ∇f in terms of $||n||_*$. To control the oscillation of f in terms of $||n||_*$, we need a Poincaré inequality. This we shall be able to get using the information about M that we derive from $||n||_*$ and $\theta(R)$ being sufficiently small.

We shall prove the Poincaré inequality we need in §1, and then prove (0.4) and (0.5) in §2.

I am grateful to Peter Jones for repeatedly asking if the theorem was true, despite my repeated protestations that it could not possibly be.

1. Poincaré inequality

Lemma 1.1. *If $\|n\|_*$ and $\theta(R)$ are sufficiently small, then for any smooth function f on M, any $z \in M$, and any $\rho \in (0, \frac{1}{200}R]$, we have*

$$\frac{1}{\rho^{d+1}} \int_{B(z,\rho)\cap M} |f(x) - f_{z,\rho}|\, dx \leq A\left(\frac{1}{\rho^d} \int_{B(z,2\rho)} |\nabla_T f(y)|^2\, dy\right)^{1/2}. \tag{1.2}$$

Here $\nabla_T f$ denotes the tangential gradient of f, $f_{z,\rho}$ denotes the mean of f over $M \cap B(z, \rho)$, and A depends only on the dimension.

One can of course prove sharper versions of this, e.g., where the 2 on the right side of (1.2) is replaced by a $1+\delta$, perhaps even by 1. One can also derive Sobolev embedding theorems on chord-arc surfaces with small constant from (1.2) and its sharper versions using standard methods. (This uses the fact that for these M's we have $|B(x, \rho) \cap M| \approx \rho^d$ if $x \in M$.)

To prove Lemma 1.1, we need to know how the geometry of M is controlled by the assumptions that $\|n\|_*$ and $\theta(R)$ are small. Let $x \in M$ and $\rho \leq \frac{1}{100}R$ be given. As in [Se1] (especially §§5 and 3), $M \cap B(x, \rho)$ must be well approximated by a Lipschitz graph if $\|n\|_*$ and $\theta(R)$ are small enough, in the following ways.

Set $\gamma = \|n\|_* + \theta(R)$. (In [Se1], γ denotes this quantity with $R = \infty$.) If γ is small enough, then for all $z \in M$, $0 < r < \frac{1}{100}R$ we have that

$$(1 - C\gamma)\nu_d r^d \leq |M \cap B(z, r)| \leq \left(1 + C\gamma \log \tfrac{1}{\gamma}\right)\nu_d r^d, \tag{1.3}$$

where ν_d denotes the volume of the unit ball in $\mathbf{R}^d$.

Put $n_0 = n_{x,2\rho}$, and let $\mu \in (10\gamma, \frac{1}{3})$ be arbitrary. Think of μ as small, but large compared to γ. Let H_0 denote the hyperplane through x and normal to n_0. We can identify $\mathbf{R}^{d+1}$ with $H_0 \times \mathbf{R}$, with $(\xi, t) \in H_0 \times \mathbf{R}$ corresponding to $\xi + tn_0 \in \mathbf{R}^{d+1}$. Let $\Pi: \mathbf{R}^{d+1} \to H_0$ denote the obvious orthogonal projection, and let $\mathscr{C}$ denote the soup can $\{(\xi, t): |\xi - x| \leq \rho,\ |t| \leq \rho\}$. Thus $B(x, \rho) \subseteq \mathscr{C} \subseteq B(x, 2\rho)$.

There is a Lipschitz function $g: H_0 \to \mathbf{R}$, $\|\nabla g\|_\infty \leq C\mu$, whose graph $G = \{(\xi, t): g(\xi) = t\}$ approximates $M \cap \mathscr{C}$ in the following ways. First,

$$|\mathscr{C} \cap \{(M\backslash G) \cup (G\backslash M)\}| \leq C \exp(-a\mu\gamma^{-1})\rho^d$$

for some a, $C > 0$. Second, $\mathscr{C} \cap M = F \cup B$, where $F \subseteq G$, F is closed, $|n(z) - n_{x,\rho}| \leq C\mu$ if $z \in F$,

$$|B| \leq C \exp(-a\mu\gamma^{-1})\rho^d, \tag{1.4}$$

and if $y \in \mathscr{C} \cap M$, $y \in B$, then

$$|y - (\Pi(y), g(\Pi(y)))| \leq C\mu\, \mathrm{dist}(\Pi(y), \Pi(F)). \tag{1.5}$$

In other words, this last condition says that even the set $\mathscr{C} \cap (M \backslash G)$ stays close to G. Also, $\Pi(\mathscr{C} \cap M) = \{\xi \in H_0 : |\xi - x| \leq \rho\}$.

The proof of these facts is the same as the proof of Proposition 5.1 in [Se1]. (That result is stated for the case where $\theta(\infty)$ is small.)

The plan of the proof of Lemma 1.1 is as follows. Fix M and $R > 0$. Our a priori assumptions imply that (1.2) holds for all $\rho \in (0, \frac{1}{200}R]$ with a constant A that does not depend on f, x, or ρ but does depend on M. Let $A(M)$ denote the best such constant. We want to bound $A(M)$ by a constant that depends only on the dimension. To do this we will try to get the estimate (1.2) by approximating M by a Lipschitz graph. This will introduce certain errors that depend on $A(M)$ but which are small, from which we derive an a priori bound on $A(M)$ that does the job.

Let z, ρ, and f be given, as above. Our a priori smoothness assumptions imply that f is Lipschitz: $|f(x) - f(y)| \leq C(f)|x-y|$ if $x, y \in M$, $|x-y| \leq 1$, for some $C(f) < \infty$.

Let G be as above (with x replaced by z and ρ by 2ρ). We can identify $G \cap \mathscr{C}$ with $B(z, \rho) \cap H_0$ in the obvious way. We want to get an approximate version of (1.2) by replacing f by a model h on $B(z, \rho) \cap H_0$ and applying the standard Poincaré inequality there. Actually, it will be useful for us to define h on $B(z, \frac{3}{2}\rho) \cap H_0$.

For the rest of the proof let B_1, B_2, and B_3 denote the intersection of H_0 with $B(z, \rho)$, $B(z, \frac{3}{2}\rho)$, and $B(z, 2\rho)$, respectively. It is important to keep in mind that the B_i's live in the H_0, not $\mathbf{R}^{d+1}$.

We want to define h on B_2. On $B_2 \cap \Pi(F)$ we set $h(\xi) = f((\xi, g(\xi)))$. To define h on $B_2 \backslash \Pi(F)$, we use a Whitney-type construction. (See [St, Chapter 6].) Let $\{Q_i\}$ be a Whitney decomposition of $H_0 \backslash \Pi(F)$ inside of H_0: each Q_i is a cube, the Q_i's are pairwise disjoint, $\bigcup Q_i = H_0 \backslash \Pi(F)$, and $\operatorname{diam} Q_i$ is comparable to $\operatorname{dist}(Q_i, \Pi(F))$. By subdividing the Q_i's if necessary, we can also assume that

$$\operatorname{diam}(Q_i) \leq \frac{1}{500} \operatorname{dist}(Q_i, \Pi(F)). \tag{1.6}$$

As on pp. 169–170 of [St] we can associate to the Q_i's a partition of unity $\{\varphi_i\}$; each φ_i is supported in $Q_i^* = \frac{11}{10} Q_i$, $\sum \varphi_i = 1$ on $H_0 \backslash \Pi(F)$, and $|\nabla \varphi_i| \leq C D_i^{-1}$, where D_i denotes the diameter of Q_i.

Let I denote the set of i's such that $Q_i^* \cap B_2 \neq \varnothing$. Because $\Pi(F) \cap B_2 \neq \varnothing$, we can conclude from (1.6) that $D_i \leq \frac{1}{100}\rho$, and in particular that $Q_i^* \subseteq B_3$. Thus we obtain

$$\sum_{i \in I} D_i^d \leq C \sum_{i \in I} |Q_i| = C \left| \bigcup_{i \in I} Q_i \right| \leq C |B_3 \backslash \Pi(F)|. \tag{1.7}$$

If γ is small, we can make this last be small compared to ρ^d using (1.4).

For each $i \in I$ let q_i be a point in $M \cap \mathscr{C}$ such that $\Pi(q_i)$ is the center of Q_i. Let f_i denote the mean value of f over $M \cap B(q_i, D_i)$. Define h on

$B_2\backslash\Pi(F)$ by

$$h(\xi) = \sum_{i\in I} f_i\varphi_i(\xi).$$

Standard arguments imply that h is Lipschitz on B_2, because f is. ((1.5) is helpful in this regard.) We want to control ∇h in term of $\nabla_T f$.

We first observe that

$$|\nabla h(\xi)| \leq C|\nabla_T f(\xi, q(\xi))| \quad \text{a.e. on } B_2\cap\Pi(F). \tag{1.8}$$

To see this, we first consider, for each $\xi \in \Pi(F)$,

$$\limsup_{t\to 0^+}\left(\sup\frac{D_i}{t}\right),$$

where the supremum is taken over those $i \in I$ such that $Q_i \subseteq B(\xi, t)$. One can easily check that the set of ξ's in $\Pi(F)$ for which the limit is positive has measure zero, because they cannot be points of density of $\Pi(F)$. When the limit is zero, it is fairly easy to prove (1.8), by computing ∇h exactly in terms of $\nabla_T f$. (Here it is helpful to recall that M is a graph over H_0 in a small neighborhood about $\xi \in \Pi(F)$, because of our a priori smoothness assumptions on M.)

Now suppose that $\xi \in B_2\backslash\Pi(F)$. Fix $j \in I$ so that $\xi \in Q_j$. Then

$$\begin{aligned}
|\nabla h(\xi)| &\leq C\max_{\substack{i\in I\\ \xi\in Q_i^*}} |f_i - f_j|D_j^{-1}\\
&\leq CD_j^{-d-1}\int_{B(q_j, 20D_j)} |f - f_j|\\
&\leq CA(M)\left(\frac{1}{D_j^d}\int_{B(q_j, 40D_j)} |\nabla_T f|^2\right)^{1/2}.
\end{aligned}$$

The second inequality uses arguments as on p. 169 of [St] to conclude that if $Q_j \cap Q_i^* \neq \varnothing$, then $\operatorname{diam} Q_i \leq 4\operatorname{diam} Q_j$. The last inequality uses (1.2) with $A = A(M)$ and z, ρ replaced by q_j, $20D_j$.

Combining these estimates, and then applying Cauchy-Schwarz, we obtain

$$\begin{aligned}
\int_{B_2} |\nabla h| &\leq C\int_{B(z, 2\rho)\cap M} |\nabla_T f| + CA(M)\sum_{j\in I}|Q_j|\left(D_j^{-d}\int_{B(q_j, 40D_j)\cap M}|\nabla_T f|^2\right)^{1/2}\\
&\leq C\int_{B(z, 2\rho)\cap M} |\nabla_T f| + CA(M)\left(\sum_{j\in I} D_j^d\right)^{1/2}\left(\int_{B(z, 2\rho)\cap M}|\nabla_T f|^2\right)^{1/2}.
\end{aligned}$$

In the second step we used the fact that the balls $B(q_j, 40D_j)$ have bounded overlap. ((1.6) is relevant here.) From (1.7) we get

$$\rho^{-d}\int_{B_2} |\nabla h| \leq C[1 + A(M)(\rho^{-d}|B_3\backslash\Pi(F)|)^{1/2}]\left(\rho^{-d}\int_{B(z, 2\rho)\cap M}|\nabla_T f|^2\right)^{1/2}. \tag{1.9}$$

The standard Poincaré inequality now yields

$$\rho^{-d-1}\int_{B_2}|h-h_{B_2}| \le C[1+A(M)(\rho^{-d}|B_3\backslash\Pi(F)|)^{1/2}]\left(\rho^{-d}\int_{B(z,2\rho)\cap M}|\nabla_T f|^2\right)^{1/2},$$

where h_{B_2} denotes the average of h over B_2.

We want to use this to estimate the left side of (1.2). The idea is that h is a good model for f. To effect this idea it is better to work with the function $\widetilde{h}$ on $M\cap B(z,\rho)$ defined by $\widetilde{h}(z)=h(\Pi(z))$.

We need to estimate $\int_{B(z,\rho)\cap M}|\widetilde{h}-h_{B_2}|$. For trivial reasons we have that

$$\int_{B(z,\rho)\cap F}|\widetilde{h}-h_{B_2}| \le C\int_{B_2\cap\Pi(F)}|h-h_{B_2}|.$$

For the remainder, we compute as follows:

$$\begin{aligned}\int_{B(z,\rho)\cap(M\backslash F)}|\widetilde{h}-h_{B_2}| &\le \sum_i \int_{B(z,\rho)\cap M\cap\Pi^{-1}(Q_i)}|\widetilde{h}-h_{B_2}| \\ &\le C\sum_{Q_i\cap B_1\neq\varnothing} D_i^d \sup_{Q_i}|h-h_{B_2}|.\end{aligned}$$

That $|B(z,\rho)\cap M\cap\Pi^{-1}(Q_i)| \le CD_i^d$ follows from (1.5), (1.3), and the fact that $\operatorname{dist}(Q_i,\Pi(F))\approx \operatorname{diam} Q_i$. From here we get

$$\int_{B(z,\rho)\cap(M\backslash F)}|\widetilde{h}-h_{B_2}| \le C\sum_{Q_i\cap B_1\neq\varnothing}\left\{\left(\int_{Q_i}|h-h_{B_2}|\right)+D_i^{d+1}\sup_{Q_i}|\nabla h|\right\}.$$

Thus

$$\begin{aligned}(1.10)\quad \int_{B(z,\rho)\cap M}|\widetilde{h}-h_{B_2}| &\le C\int_{B_2}|h-h_{B_2}|+C\sum_{Q_i\subseteq B_2}D_i^{d+1}\sup_{Q_i}|\nabla h| \\ &\le C\rho^{d+1}[1+A(M)(\rho^d|B_3\backslash\Pi(F)|)^{1/2}]\left(\rho^d\int_{B(z,2\rho)\cap M}|\nabla_T f|^2\right)^{1/2}.\end{aligned}$$

For the first inequality, we used the fact that $Q_i\subseteq B_2$ if $Q_i\cap B_i\neq\varnothing$. For the second, we used estimates for $|\nabla h|$ exactly like those used in proving (1.9).

Now we want to estimate $f-\widetilde{h}$. By definitions $f=\widetilde{h}$ on F, and so

$$\int_{B(z,\rho)\cap M}|f-\widetilde{h}| \le \sum_i\int_{B(z,\rho)\cap M\cap\Pi^{-1}(Q_i)}\{|f-f_i|+|\widetilde{h}-f_i|\}.$$

By (1.5), $B(z,\rho)\cap M\cap\Pi^{-1}(Q_i)\subseteq B(q_i,2D_i)$ if μ is chosen small enough. Thus we have

$$\int_{B(z,\rho)\cap M}|f-\widetilde{h}| \le \sum_{i\in I}\int_{B(q_i,2D_i)\cap M}|f-f_i| + C\sum_{Q_i\subseteq B_2}D_i^d\sup_{Q_i}|h-f_i|.$$

By construction, if $Q_i\subseteq B_2$, then

$$\sup_{Q_i}|h-f_i| \le C\max_{j:Q_j^*\cap Q_i\neq\varnothing}|f_i-f_j|.$$

Using these inequalities, one can show that

$$\begin{aligned}\rho^{-d-1}&\int_{B(z,\rho)\cap M}|f-\widetilde{h}| \\ &\le CA(M)(\rho^{-d}|B_3\backslash\Pi(F)|)^{1/2}\left(\rho^{-d}\int_{B(z,2\rho)\cap M}|\nabla_T f|^2\right)^{1/2}\end{aligned}\tag{1.11}$$

using the same calculations as used to prove (1.9). Combining (1.10) and (1.11) gives

$$\begin{aligned}\rho^{-d-1}&\int_{B(z,\rho)\cap M}|f-h_{B_2}| \\ &\le C[1+A(M)(\rho^{-d}|B_3\backslash\Pi(F)|)^{1/2}]\left(\rho^{-d}\int_{B(z,2\rho)\cap M}|\nabla_T f|^2\right)^{1/2}.\end{aligned}$$

Of course h_{B_2} is just a constant, and if we replace it by $f_{z,\rho}$ in the left-hand side, then we can only increase the total by a bounded factor, by a standard computation.

Since $A(M)$ is by definition the best constant in (1.2) for M, we conclude now that

$$A(M)\le C[1+A(M)(\rho^{-d}|B_3\backslash\Pi(F)|)^{1/2}].$$

By taking γ small enough, we can choose μ so that μ is small and so that

$$C(\rho^{-d}|B_3\backslash\Pi(F)|)^{1/2}\le\tfrac{1}{2},$$

because of (1.4). This gives $A(M)\le C+\frac{1}{2}A(M)$, which implies that $A(M)\le C$, as desired. This completes the proof of Lemma 1.1.

2. Proof of the theorem

We need to prove (0.4) and (0.5). Let us begin with (0.4).

Fix $R>0$. If $\theta(R)$ is small enough, then from (1.3) we get that

$$C^{-1}r^d\le|M'\cap B(z,r)|\le Cr^d\quad\text{if } 0<r\le 10R.$$

From this and the John-Nirenberg lemma for BMO on spaces of homogeneous type [CW] we conclude that for any $p<\infty$,

$$\sup_{\rho\le 5R}\sup_{x\in M}\left(\frac{1}{\rho}\int_{B(x,\rho)\cap M}|n(y)-n_{x,\rho}|^p\,dy\right)^{1/p}\le C(p)\|n\|_*.\tag{2.2}$$

Let us estimate $\theta(\frac{1}{400}R)$. Fix $p\le\frac{1}{400}R$ and $x\in M$. Set $f(y)=\langle x-y,n_{x,\rho}\rangle$, so that $|\nabla_T f(y)|\le|n(y)-n_{x,\rho}|$. From (1.2) we have that for $0 < s < \rho$, $z \in B(x, \rho) \cap M$,

$$\begin{aligned}
&\frac{1}{s^{d+1}} \int_{B(z,s)\cap M} |f(y) - f_{z,s}|\, dy \\
&\leq C\left(\frac{1}{s^d}\int_{B(z,2s)\cap M} |\nabla_T f|^2\right)^{1/2} \leq Cs^{-d/2}\left(\int_{B(z,2s)\cap M} |n(y) - n_{x,\rho}|^2\right)^{1/2} \\
&\leq Cs^{-d/2}\left(\int_{B(z,2s)\cap M} |n(y) - n_{x,\rho}|^{4d}\right)^{1/4d} s^{(2d-1)/4} \\
&\leq Cs^{-1/4}\rho^{1/4}\left(\frac{1}{\rho^d}\int_{B(x,3\rho)\cap M} |n(y) - n_{x,\rho}|^{4d}\right)^{1/4d} \\
&\leq Cs^{-1/4}\rho^{1/4}\|n\|_*.
\end{aligned}$$

From these estimates and standard reasoning we get

$$\sup_{y\in M\cap B(x,\rho)} |f(y) - f(x)|\rho^{-1} \leq C\|n\|_*,$$

which implies (0.4).

To prove (0.5), we first need some preliminaries. Let Ω_+ and Ω_- denote the two complementary components of M inside $\mathbf{R}^{d+1}$, with $n(x)$ pointing into Ω_+. Given $x \in M$, set

$$B_+(x, R) = \{y \in B(x, R) \colon \langle y - x, n_{x,r}\rangle > R\theta(R)\},$$

and define $B_-(x, R)$ similarly, with $> R\theta(R)$ replaced by $< -R\theta(R)$. Then $B_+(x, R) \subseteq \Omega_+$, and $B_-(x, R) \subseteq \Omega_-$.

To see this, let F be as in §1 (discussed near (1.5)), with $\rho = \frac{1}{100}R$. Let $z \in F$ be arbitrary. From (1.5) and the fact that $|n(z) - n_{x,\rho}| \leq C\mu$ we get

$$\{z + tn(z) \colon 0 < t \leq \rho\} \cap M = \varnothing,$$

at least if we chose μ small enough. For t sufficiently small, $z + tn(z) \in \Omega_+$, by definitions. Hence $z + \rho n(z) \in \Omega_+$, and since $B_+(x, R) \cap M = \varnothing$, we get that $B_+(x, R) \subseteq M$. The other case is similar.

Let us now prove (0.5). Let $x, z \in M$ be given, with $|z - x| \leq \frac{3}{2}R$. From the preceding we obtain

$$B_+(z, R) \cap B_-(x, R) = \varnothing, \qquad B_-(z, R) \cap B_+(x, R) = \varnothing.$$

It is easy to use this to control $\langle z - x, n_{x,R}\rangle$ and thereby prove (0.5).

This finishes the proof of the theorem.

References

[CW] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.

[Se1] S. Semmes, *Chord-arc surfaces with small constant.* I, Adv. in Math. (to appear).

[Se2] ——, *Chord-arc surfaces with small constant.* II: *Good parameterizations*, Adv. in Math. (to appear).

[Se3] ——, *The Cauchy integral, chord-arc curves, and quasiconformal mappings*, The Bieberback Conjecture (Proc. Sympos. on the Occasion of the Proof) (A. Baernstein et al., eds.), Math. Surveys Monographs, no. 21, Amer. Math. Soc., Providence, RI, 1986, pp. 167–183.

[Si] L. Simon, *Lectures on geometric measure theory*, Proc. Centre Math. Anal., Australian National University, 1984.

[St] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1970.

DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TEXAS 77251