BERNSTEIN-TYPE INEQUALITIES FOR THE DERIVATIVES OF CONSTRAINED POLYNOMIALS

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Abstract. Generalizing a number of earlier results, P. Borwein established a sharp Markov-type inequality on \([-1,1]\) for the derivatives of polynomials \(p \in \pi_n\) having at most \(k\) (0 \(\leq k \leq n\)) zeros in the complex unit disk. Using Lorentz representation and a Markov-type inequality for the derivative of Müntz polynomials due to D. Newman, we give a surprisingly short proof of Borwein's Theorem. The new result of this paper is to obtain a sharp Bernstein-type analogue of Borwein's Theorem. By the same method we prove a sharp Bernstein-type inequality for another wide family of classes of constrained polynomials.

1. Introduction, notations

Markov's inequality, which plays a significant role in approximation theory and related areas, states that

\[ \max_{-1 \leq x \leq 1} |p'(x)| \leq n^2 \max_{-1 \leq x \leq 1} |p(x)| \]

for every polynomial \(p \in \pi_n\), where \(\pi_n\) denotes the set of all real algebraic polynomials of degree at most \(n\). The pointwise algebraic Bernstein-type analogue asserts that

\[ |p'(y)| \leq \frac{n}{\sqrt{1 - y^2}} \max_{-1 \leq x \leq 1} |p(x)| \quad (-1 < y < 1) \]

for every polynomial \(p \in \pi_n\). On every fixed subinterval \([-a, a]\) (0 < \(a < 1\)), (2) gives a much better upper bound than (1). Let \(S_n^k(z, r)\) be the family of polynomials from \(\pi_n\) which have at most \(k\) zeros in the open disk of the complex plane with center \(z\) and radius \(r\). A number of papers were written on Markov- and Bernstein-type inequalities for the derivatives of polynomials from \(S_n^k(0, 1)\) in certain special cases. When \(k = 0\), see [5], [6], and [9]; when \(k\) is small compared with \(n\), [7] and [11] give reasonable results. Finally, proving J. Szabados's conjecture, P. Borwein [1] verified the following sharp inequality:

\[ \max_{-1 \leq x \leq 1} |p'(x)| \leq n \max_{-1 \leq x \leq 1} |p(x)| \]

\(\leq \frac{n}{\sqrt{1 - y^2}} \max_{-1 \leq x \leq 1} |p(x)| \quad (-1 < y < 1) \)

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829
Theorem 1. We have
\[
\max_{0 \leq x \leq 1} |p'(x)| \leq cn(k + 1) \max_{0 \leq x \leq 1} |p(x)|
\]
for every \( p \in S_n^k(1/2, 1/2) \) with some absolute constant \( c \leq 18 \).

Using a Lorentz representation of a polynomial from \( S_n^0(1/2, 1/2) \) and a Markov-type inequality for the derivative of Müntz polynomials, we will present a very short proof of Theorem 1. A sharp Markov-type inequality was established in [2] for another family of classes of constrained polynomials.

Theorem 2. We have
\[
\max_{-1 \leq x \leq 1} |p'(x)| \leq \min \left\{ n^2, \frac{cn}{\sqrt{r}} \right\} \max_{-1 \leq x \leq 1} |p(x)|
\]
for every polynomial from \( \pi_n \) having no zeros in the open disks with diameters \([-1, -1 + 2r]\) and \([1 - 2r, 1]\), respectively, where \( 0 < r \leq 1 \) and \( c \leq 20 \) is an absolute constant.

For \( 0 < r \leq 1 \), we define
\[
K(r) = \bigcup_{a \in [-1+r, 1-r]} \{ z \in \mathbb{C} : |z - a| < r \}
\]
and denote by \( W_n^0(r) \) the set of those polynomials from \( \pi_n \) which have no zeros in \( K(r) \). The main goal of this paper is to obtain sharp Bernstein-type inequalities for the derivatives of polynomials from \( S_n^k(0, 1) \) and \( W_n^0(r) \), respectively.

2. NEW RESULTS

We will prove the following Bernstein-type inequalities:

Theorem 3. We have
\[
|p'(y)| \leq c \frac{\sqrt{n(k + 1)}}{1 - y^2} \max_{-1 \leq x \leq 1} |p(x)| \quad (-1 < y < 1)
\]
for every \( p \in S_n^k(0, 1) \), where \( c \) is an absolute constant.

Theorem 4. We have
\[
\begin{align*}
(i) \quad |p'(y)| &\leq c \frac{n}{r(1 - y^2)} \max_{-1 \leq x \leq 1} |p(x)| \quad (-1 + r < y < 1 - r) \\
(ii) \quad |p'(y)| &\leq c \frac{1}{r(1 - y^2)} \max_{-1 \leq x \leq 1} |p(x)| \quad (-1 < y < 1)
\end{align*}
\]
for every polynomial \( p \in W_n^0(r) \) \( (0 < r \leq 1) \) with certain absolute constants \( c \).
3. THE SHARPNESS OF OUR THEOREMS

The sharpness of Theorem 1 was proved by J. Szabados [10]. It was shown in [3] that

\[ \sup_{p \in S^k_0(0, 1)} \frac{|p'(0)|}{\max_{-1 \leq x \leq 1} |p(x)|} \geq c \sqrt{n(k + 1)} \]

with some absolute constant \( c > 0 \).

**Conjecture 1.** The pointwise factor \((1 - y^2)^{-1}\) in Theorem 3 can be replaced by \((1 - y^2)^{-1/2}\).

**Conjecture 2.** We have

\[ \sup_{p \in W^r_n} \frac{|p'(0)|}{\max_{-1 \leq x \leq 1} |p(x)|} \geq c \sqrt{\frac{n}{r}} \left( \frac{1}{n} \leq r \leq 1 \right) \]

with some absolute constant \( c > 0 \).

**Remark 1.** With J. Szabados [4], we proved that

\[ |p'(y)| \leq c \frac{\sqrt{n(k + 1)^2}}{\sqrt{1 - y^2}} \max_{-1 \leq x \leq 1} |p(x)| \quad (-1 < y < 1) \]

for every \( p \in S^k_0(0, 1) \), where \( c \) is an absolute constant.

4. A NEW PROOF OF BORWEIN'S THEOREM

In this section we give a short proof of Theorem 1. Let \( \Lambda = \{\lambda_j\}_{j=1}^N \) be an increasing set of positive numbers. Denote by \( \pi(\Lambda) \) the collection of \( \Lambda \) polynomials of the form

\[ p(x) = a_0 + \sum_{j=1}^N a_j x^j \quad (0 \leq x < \infty) \tag{3} \]

with real coefficients \( a_j \). We will use the following Markov-type theorem for the derivative of \( \Lambda \) polynomials.

**Theorem 5 (D. Newman).** For every \( \Lambda \) polynomial \( p \) of type (3), we have

\[ \frac{2}{3} \sum_{j=1}^N \lambda_j \leq \sup_{\pi(\Lambda)} \frac{\max_{0 \leq x \leq 1} |p'(x)x|}{\max_{0 \leq x \leq 1} |p(x)|} \leq 11 \sum_{j=1}^N \lambda_j. \]

The proof of Theorem 5 may be found in [8]. As a straightforward consequence of Theorem 5, we obtain the following:

**Proposition.** Let \( p(x) = x^{n-k} Q_k(x) \), where \( Q_k \in \pi_k \). Then

\[ |p'(1)| \leq 11n(k + 1) \max_{0 \leq x \leq 1} |p(x)|. \]
Proof of the proposition. By Theorem 5 we have

$$|p'(1)| \leq 11 \left( \sum_{j=n-k}^{n} j \right) \max_{0 \leq x \leq 1} |p(x)| \leq 11n(k + 1) \max_{0 \leq x \leq 1} |p(x)|. \quad \Box$$

Now let $p \in S^{k}_{n}(1/2, 1/2)$. By an observation of G. G. Lorentz [9], we have $p = wQ_{k}$, where $Q_{k} \in \pi_{k}$ and

$$w(x) = \sum_{j=0}^{n-k} a_{j}(1-x)^{j}x^{n-k-j}, \quad \text{with all } a_{j} \geq 0.$$  

We may assume that $n - k \geq 1$; otherwise, (1) gives Theorem 1. Using the Proposition and $a_{j} \geq 0$ ($0 \leq j \leq n - k$), we obtain

$$|p'(1)| = |(a_{0}x^{n-k}Q_{k}(x))'(1) + (a_{1}(1-x)x^{n-k-1}Q_{k}(x))'(1)|$$

$$= |(x^{n-k-1}(a_{0}xQ_{k}(x) + a_{1}(1-x)Q_{k}(x)))'(1)|$$

$$\leq 11n(k + 2) \max_{0 \leq x \leq 1} \sum_{j=0}^{1} a_{j}(1-x)^{j}x^{n-k-j}Q_{k}(x)$$

$$\leq 11n(k + 2) \max_{0 \leq x \leq 1} \sum_{j=0}^{n-k} a_{j}(1-x)^{j}x^{n-k-j}Q_{k}(x)$$

$$= 11n(k + 2) \max_{0 \leq x \leq 1} |p(x)|. \tag{4}$$

Now let $y \in [0, 1]$ be arbitrary. To estimate $|p'(y)|$, we may assume that $1/2 \leq y \leq 1$; otherwise, $P(x) = p(1-x)$ is $S^{k}_{n}(1/2, 1/2)$ can be studied. If $p \in S^{k}_{n}(1/2, 1/2)$, then $p \in S^{k}_{n}(y/2, y/2)$; hence, by a linear transformation, (4) yields

$$|p'(y)| \leq \frac{11}{y}n(k + 2) \max_{0 \leq x \leq y} |p(x)| \leq 22n(k + 2) \max_{0 \leq x \leq y} |p(x)|,$$

which finishes the proof of Borwein’s Theorem. \quad \Box

Remark 2. Observe that $p \in S^{k}_{n}(1/2, 1/2)$ does not necessarily imply $p' \in S^{k}_{n}(1/2, 1/2)$; therefore, the generalization of Borwein’s inequality for higher derivatives does not follow immediately from the case of the first derivative. Nevertheless, the inequality

$$\max_{0 \leq x \leq 1} |p^{(m)}(x)| \leq c(m)(n(k + 1))^{m} \max_{0 \leq x \leq 1} |p(x)|$$

can be proved about as briefly, relying only on a Lorentz representation of $p \in S^{k}_{n}(1/2, 1/2)$ and Newman’s inequality.

5. Lemmas for Theorem 3

It is sufficient to prove Theorem 3 when $y = 0$, since from this we will obtain the statement in the general case by a linear transformation. By our first lemma, we introduce an extremal polynomial $Q \in S^{k}_{n}(0, 1)$. 

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Lemma 1. For every \( n \) and \( k \) (\( 0 \leq k \leq n \)) natural numbers, there exists a polynomial \( Q \in S_n^k(0, 1) \) such that

\[
\frac{|Q'(0)|}{\max_{-1 \leq x \leq 1} |Q(x)|} = \sup_{p \in S_n^k(0, 1)} \frac{|p'(0)|}{\max_{-1 \leq x \leq 1} |p(x)|}.
\]

The next lemma gives some information on the zeros of the extremal polynomial \( Q \) introduced by Lemma 1.

Lemma 2. Let \( Q \) be defined by Lemma 1. Then \( Q \) has only real zeros, and at most \( k + 1 \) of them are different from \( \pm 1 \) (counting multiplicities).

From Theorem 1 we will easily deduce

Lemma 3. Let \( \delta = (36n(k+1))^{-1} \). Then

\[
\max_{-\delta \leq x \leq \delta} |q(x)| < 2 \max_{0 \leq x \leq 1} |q(x)|
\]

for every \( q \in S_n^k(1/2, 1/2) \) having all its zeros in \([0, \infty)\).

From Lemma 3 we will easily obtain

Lemma 4. Let \( z_0 = i(36n(k+1))^{-1/2} \) where \( i \) is the imaginary unit. Then

\[
|p(z_0)| < \sqrt{2} \max_{-1 \leq x \leq 1} |p(x)|
\]

for every \( p \in S_n^k(0, 1) \) having only real zeros.

Remark 3. The inequalities of Lemmas 3 and 4 can be proved for all \( p \in S_n^k(1/2, 1/2) \) and \( p \in S_n^k(0, 1) \), respectively, but under our additional assumptions their proofs are much shorter.

We will prove Theorem 3 by Cauchy’s integral formula and these lemmas.

6. Lemmas for Theorem 4

The way to prove Theorem 4 is very similar to the previous section. We introduce an extremal polynomial by

Lemma 5. For every \( n \) natural and \( 0 < r \leq 1 \) real numbers, there exists a polynomial \( Q \in S_n^0(0, r) \) (\( 0 < r \leq 1 \)) such that

\[
\frac{|Q'(0)|}{\max_{-1 \leq x \leq 1} |Q(x)|} = \sup_{p \in S_n^0(0, r)} \frac{|p'(0)|}{\max_{-1 \leq x \leq 1} |p(x)|}.
\]

By a variational method we will obtain

Lemma 6. Let \( Q \) be defined by Lemma 5. Then \( Q \) has only real zeros.

From Theorem 2 we will easily prove

Lemma 7. Let \( 0 < R \leq 1 \) and \( \delta = \sqrt{R/(8n)} \). Then

\[
\max_{-\delta \leq x \leq 1} |q(x)| < 2 \max_{0 \leq x \leq 1} |q(x)|
\]

for every \( q \in S_n^0(0, R) \) having all its zeros in \([R, \infty)\).

Our last lemma will be a straightforward consequence of Lemma 7.
Lemma 8. Let $0 < r \leq 1$ and $z_0 = i\sqrt{r/(8n)}$, where $i$ is the imaginary unit. Then

$$|p(z_0)| < \sqrt{2} \max_{-1 \leq x \leq 1} |p(x)|$$

for every $p \in S_n^0(0, r)$ having only real zeros.

Remark 4. The inequalities of Lemmas 7 and 8 can be verified for all $p \in S_n^0(0, R)$ ($0 < R \leq 1$) and $p \in S_n^0(r)$ ($0 < r \leq 1$), respectively, but under our additional assumptions their proofs are simpler.

We will prove Theorem 4 by Cauchy's integral formula, similarly to the proof of Theorem 3.

7. A Bernstein-Walsh type problem for polynomials from $S_n^k(0, 1)$

We would like some information on the magnitude of $|p(z)|$ ($z \in \mathbb{C}$) when $p \in S_n^k(0, 1)$ and $\max_{-1 \leq x \leq 1} |p(x)| = 1$.

Conjecture 3. Let $D$ be the ellipse of the complex plane with axes $[-a, a]$ and $[-b, b]$, where $a = 1 + (n(k+1))^{-1}$ and $b = i(n(k+1))^{-1/2}$ ($i$ is the imaginary unit). Then there is an absolute constant $c$ such that

$$|p(z)| \leq c \max_{-1 \leq x \leq 1} |p(x)| \quad (z \in D)$$

for every $p \in S_n^k(0, 1)$ ($0 \leq k \leq n$).

Conjecture 1 could be obtained immediately from Conjecture 3 and Cauchy's integral formula.

8. Proofs of the lemmas for Theorem 3

The proof of Lemma 1 is a straightforward application of Hurwitz's Theorem.

Proof of Lemma 2. Assume indirectly that there are at least two zeros of $Q$ outside the open unit disk and different from $\pm 1$. If $z_1$ is a nonreal zero of $Q$, then the polynomial

$$Q_\varepsilon(x) = Q(x) - \varepsilon \frac{Q(x) x^2}{(x - z_1)(x - z_1^*)} \in S_n^k(0, 1)$$

with a sufficiently small $\varepsilon > 0$ contradicts the maximality of $Q$. If $a$ and $b$ are real zeros of $Q$ such that $|a| \geq |b| > 1$, then the polynomial

$$Q_\varepsilon(x) = Q(x) - \varepsilon \text{sgn}(ab) \frac{Q(x) x^2}{(x - a)(x - b)} \in S_n^k(0, 1)$$

with a sufficiently small $\varepsilon > 0$ contradicts the maximality of $Q$. Thus Lemma 2 is proved. \(\square\)

Lemma 3 follows immediately from Theorem 1 and the Mean Value Theorem.
Proof of Lemma 3. Assume indirectly that there is a $-\delta \leq y < 0$ such that

(5) \[ |q(y)| = 2 \max_{0 \leq x \leq 1} |q(x)| \]

for a polynomial $q \in S^k_n(1/2, 1/2)$ having all its zeros in $[0, \infty)$. Then using (5), $-\delta \leq y < 0$, $\delta = (36n(k + 1))^{-1}$, the Mean Value Theorem, the monotonicity of $|q'|$ on $(-\infty, 0]$, and Theorem 1 transformed linearly to the interval $[y, 1]$, we can find a $\xi \in (y, 0)$ such that

\[ 36n(k + 1) \max_{0 \leq x \leq 1} |q(x)| \leq \left| \frac{q(y) - q(0)}{y} \right| = |q'(\xi)| \leq |q'(y)| \]

\[ \leq \frac{18}{1 - y} n(k + 1) \max_{y \leq x \leq 1} |q(x)| \]

\[ < 18n(k + 1) \max_{y \leq x \leq 1} |q(x)| \]

\[ = 36n(k + 1) \max_{0 \leq x \leq 1} |q(x)|, \]

a contradiction. \qed

Proof of Lemma 4. We may assume that $p \in S^k_n(0, 1)$ is monic; thus, let $p(x) = \prod_{j=1}^s (x - u_j)$ with some $s \leq n$. Applying Lemma 3 to the polynomial

\[ q(x) = \prod_{j=1}^s (x - u_j)^2 \in S^k_n \left( \frac{1}{2}, \frac{1}{2} \right), \]

we easily deduce that

(6) \[ |q(-(36n(k + 1))^{-1})| < 2 \max_{0 \leq x \leq 1} |q(x)| = 2 \max_{0 \leq x \leq 1} |q(x^2)| \]

\[ = 2 \max_{-1 \leq x \leq 1} |p(x)p(-x)| \leq 2 \left( \max_{-1 \leq x \leq 1} |p(x)| \right)^2. \]

Observe that

(7) \[ |p(z_0)|^2 = \prod_{j=1}^s (u_j^2 + (36n(k + 1))^{-1}) = |q(-(36n(k + 1))^{-1})|, \]

and together with (6) this yields

\[ |p(z_0)| < \sqrt{2} \max_{-1 \leq x \leq 1} |p(x)|. \]

Thus Lemma 4 is proved. \qed

9. Proofs of Lemmas for Theorem 4

The proofs of Lemmas 5–8 are very similar to the corresponding ones from §8. The proofs of Lemmas 5 and 6 are exactly the same as those of Lemmas 1 and 2; therefore, we do not give any details.
Proof of Lemma 7. We have

\[ |q'(0)| \leq \frac{4n}{\sqrt{R}} \max_{0 \leq x \leq 1} |q(x)| \]

for every polynomial \( q \in \pi_n \) having all its zeros in \([R, \infty)\). This inequality follows from the proof of Theorem 1 in [2] with a certain multiplicative constant \( c \) instead of 4. The fact that \( c = 4 \) can be chosen was pointed out by M. v. Golitschek and G. G. Lorentz. Now assume indirectly that there exists a \( -\delta \leq y < 0 \) such that

\[ |q(y)| = 2 \max_{0 \leq x \leq 1} |q(x)| \]

for a polynomial \( q \in \pi_n \) having all its zeros in \([R, \infty)\). Then (9), \( -\delta \leq y < 0 \), \( \delta = \sqrt{R}/(8n) \), the Mean Value Theorem, the monotonicity of \( |q'| \) on \((-\infty, 0]\), and (8) transformed linearly to the interval \([y, 1]\) imply that there exists a \( \xi \in (y, 0) \) such that

\[
\begin{align*}
\frac{8n}{\sqrt{R}} \max_{0 \leq x \leq 1} |q(x)| & \leq \left| \frac{q(y) - q(0)}{y} \right| = \left| q'(\xi) \right| \leq |q'(y)| \\
& \leq \frac{4n}{(1-y)\sqrt{R-y}} \max_{y \leq x \leq 1} |q(x)| \\
& < \frac{4n}{\sqrt{R}} \max_{y \leq x \leq 1} |q(x)| = \frac{8n}{\sqrt{R}} \max_{0 \leq x \leq 1} |q(x)|,
\end{align*}
\]

a contradiction. \( \square \)

Proof of Lemma 8. We may assume that \( p \in S^0_n(0, r) \) is monic; thus, let \( p(x) = \prod_{j=1}^{s} (x - u_j) \) with some \( s \leq n \). Applying Lemma 7 to the polynomial

\[ q(x) = \prod_{j=1}^{s} (x - u_j^2) \in S^0_n(0, r^2), \]

we easily deduce that

\[ \left| q \left( -\frac{\sqrt{r^2}}{8n} \right) \right| < 2 \max_{0 \leq x \leq 1} |q(x)| = 2 \max_{0 \leq x \leq 1} |q(x^2)| \]

\[ = 2 \max_{-1 \leq x \leq 1} |p(x)p(-x)| \leq 2 \left( \max_{-1 \leq x \leq 1} |p(x)| \right)^2. \]

Further,

\[ |p(z_0)|^2 = \prod_{j=1}^{s} \left( u_j^2 + \frac{r}{8n} \right) = \left| q \left( -\frac{r}{8n} \right) \right|, \]

and together with (10) this yields

\[ |p(z_0)| < \sqrt{2} \max_{-1 \leq x \leq 1} |p(x)|. \]

Thus Lemma 8 is proved. \( \square \)
10. Proofs of Theorems 3 and 4

Proof of Theorem 3. Let

\begin{equation}
D = \left\{ z \in \mathbb{C}: |\text{Re} z| \leq \frac{1}{2}, |\text{Im} z| \leq \frac{1}{12} (n(k + 1))^{-1/2} \right\}.
\end{equation}

Since \( Q \in S_n^k(0, 1) \) defined by Lemma 1 has only real zeros (see Lemma 2), from Lemma 4, by a linear transformation, we obtain

\begin{equation}
|Q(z)| \leq \sqrt{2} \max_{-1 \leq x \leq 1} |Q(x)| \quad (z \in D),
\end{equation}

where, with the notation \( y = \text{Re} z \), \([a, b] = [-1, 1 + 2y]\) if \(-\frac{1}{2} \leq y \leq 0\), and \([a, b] = [2y - 1, 1]\) if \(0 \leq y \leq \frac{1}{2}\). Now let \( S \) be the circle with center 0 and radius \((n(k + 1))^{-1/2}/12\). By Cauchy's integral formula, (12), and (13), we obtain

\begin{align*}
|Q'(0)| &= \frac{1}{2} \left| \int_{-1}^{1} \frac{Q(\xi)}{\xi^2} d\xi \right| \leq \frac{1}{2} \left| \int_{-1}^{1} \frac{Q(\xi)}{\xi^2} |d\xi| \right| \\
&\leq \frac{\pi}{12} (n(k + 1))^{-1/2} (12(n(k + 1))^{1/2})^2 \sqrt{2} \max_{-1 \leq x \leq 1} |Q(x)| \\
&\leq c \sqrt{n(k + 1)} \max_{-1 \leq x \leq 1} |Q(x)|.
\end{align*}

From this and the maximality of \( Q \), we deduce

\begin{equation}
|p'(0)| \leq c \sqrt{n(k + 1)} \max_{-1 \leq x \leq 1} |p(x)| \quad (p \in S_n^k(0, 1)),
\end{equation}

and from here we obtain the desired result in the general case \(-1 < y < 1\) by a linear transformation. Thus Theorem 3 is proved. \(\square\)

Proof of Theorem 4. We proceed in a very similar way. Let

\begin{equation}
D = \left\{ z \in \mathbb{C}: |\text{Re} z| \leq \frac{r}{2}, |\text{Im} z| \leq \sqrt{\frac{r}{32n}} \right\}.
\end{equation}

Since \( Q \in S_n^0(0, r) \) defined by Lemma 5 has only real zeros (see Lemma 6), from Lemma 8, by a linear transformation, we deduce

\begin{equation}
|Q(z)| \leq \sqrt{2} \max_{-1 \leq x \leq 1} |Q(x)| \quad (z \in D).
\end{equation}

Let \( S \) be the circle with center 0 and radius \( \sqrt{r/(32n)} \). Since \( r \geq 1/n \), we have \( \sqrt{r/(32n)} \leq r/2 \); hence, by Cauchy's integral formula, (14), and (15), we get

\begin{align*}
|Q'(0)| &= \frac{1}{2} \left| \int_{-1}^{1} \frac{Q(\xi)}{\xi^2} d\xi \right| \leq \frac{1}{2} \left| \int_{-1}^{1} \frac{Q(\xi)}{\xi^2} |d\xi| \right| \\
&\leq \pi \sqrt{\frac{r}{32n}} \sqrt{2} \max_{-1 \leq x \leq 1} |Q(x)| \leq c \sqrt{\frac{n}{r}} \max_{-1 \leq x \leq 1} |Q(x)|.
\end{align*}
Therefore the maximality of $Q$ yields

$$(17) \quad |p'(0)| \leq c \sqrt{\frac{m}{r}} \max_{-r \leq x \leq r} |p(x)| \quad (p \in S_n^0(0, r)).$$

Now observe that $p \in W_n^0(r)$ implies $p \in S_n^0(y, r)$ if $y \in [-r, r]$, and $p \in S_n^0(y, 1 - y)$ if $r < |y| < 1$. Applying (17) transformed linearly to the interval $[2y - 1, 1]$ when $0 \leq y < 1$ and $[-1, 2y + 1]$ when $-1 \leq y < 0$, we obtain (i) and (ii) immediately. Thus Theorem 4 is proved. \(\square\)

References


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