ESTIMATES FOR INVERSES OF $e^{int}$
IN SOME QUOTIENT ALGEBRAS OF $A^+$

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Abstract. We give estimates for the norm of $e^{int}$ in $A^+/I$, where $I$ is a closed ideal of $A^+$ without inner factor, provided that the hull of $I$ satisfies suitable geometric conditions.

Introduction

Let $E$ be a closed subset of the unit circle $\Gamma$ of Lebesgue measure zero, let $d\mu = \frac{1}{2\pi} d\lambda$, where $d\lambda$ is the Lebesgue measure, and let $d$ be the metric defined by $d(e^{i\theta}, e^{i\theta'}) = |\theta - \theta'|/2\pi(0, \theta, \theta' \in [0, 2\pi])$.

Let $A$ be the Wiener algebra such that

$$\int g \in L^1([0,2\pi]) \text{ such that } \sum_{n \in \mathbb{Z}} |\hat{f}(n)| < +\infty,$$

where $\|f\| = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|$, and let $A^+$ be the subalgebra of $A$ of functions in $A$ such that $f(n) = 0$ for $n < 0$. We say that $E$ is a $ZA^+$ set if there exists a nonzero function in $A^+$ vanishing on $E$; we denote by $I_E^+$ the ideal of a function in $A^+$ vanishing on $E$ and by $A^+(E)$ the quotient algebra $A^+/I_E^+(E)$.

It is shown in [2] that if $E$ is a $ZA^+$ set we have

$$\|e^{-int}\|_{A^+(E)} = 0(\exp \varepsilon \sqrt{n}) \quad \text{for every } \varepsilon > 0.$$  

Kahane and Katznelson prove in [5] that, for every $\beta > 0$, there exists a closed set $E \subset \Gamma$ such that

$$\int_0^{2\pi} \log \frac{1}{d(e^{it}, E)} dt < +\infty$$

and

$$\lim_{n \to +\infty} \inf \frac{\log(\|e^{-int}\|_{A^+(E)})}{n^{1/2}} > 0.$$  

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In this paper, we prove that, for $1 < p \leq 2$, if $E$ satisfies

$$\int_0^{2\pi} \log^p \frac{1}{d(e^{it}, E)} dt < +\infty,$$

then there exists a $c > 0$ such that

$$\|e^{-int}\|_{A^+} = O(c \exp cn^{1/(1+p)}).$$

If $I$ is a closed ideal of $A^+$ and the only common inner factor of the functions in $I$ is 1, then, if $h(I) \subset \Gamma$ and satisfies (1), where

$$h(I) = \{z \in \mathcal{D} \text{ such that } f(z) = 0 \text{ for every } f \in I\},$$

the same estimates hold. Let $E$ be a $ZA^+$ set and set $w_n = \|e^{-int}\|_{A^+_n}$. We write $E^c = \bigcup_{\nu \in N} \alpha_\nu, \beta_\nu I$ and we set $L_\nu = \mu(\alpha_\nu, \beta_\nu I) = |\beta_\nu - \alpha_\nu|/2\pi$. Let $\varphi$ and $\psi$ be defined for $t \in [0, 1]$ by:

$$\varphi(t) = m(E_t), \text{ where } E_t = \{e^{it} \in \Gamma \text{ such that } d(e^{it}, E) \leq t\},$$

$$\psi(t) = \sum_{L_\nu \leq t} L_\nu.$$

We denote by $N^e$ the smallest integer $N$ such that there exists a collection of arcs $(I_\alpha)_{\alpha \leq N}$ of measure $e$ centered on elements of $E$ and satisfying $E \subset \bigcup_{\alpha \leq N} I_\alpha$.

We begin by remarking that certain conditions on functions $\psi, \varphi$, and $N^e$ are equivalent to (1).

**Proposition 1.** Let $p \geq 1$. The following are equivalent.

(a) $a = \sum_{\nu \in N} L_\nu \log^p \frac{1}{L_\nu} < +\infty$,

(b) $b = \int_0^1 \frac{\varphi(t)}{t} \log^{p-1} \frac{1}{t} dt < +\infty$,

(c) $c = \int_0^1 \frac{\psi(t)}{t} \log^{p-1} \frac{1}{t} dt < +\infty$,

(d) $d = \int_0^{2\pi} \log^p \frac{1}{d(e^{it}, E)} dt < +\infty$.

For $p = 1$ see [3] and [6]. The proof is similar for $p > 1$.

We now give another proposition, due to Atzmon, to be used in the proof of Theorem 3.

**Proposition 2.** Let $f \in A^+$ and $\lambda \in D = \{z \in \mathbb{C} : |z| < 1\}$. Let $\Phi$ be defined by $\Phi(f, \lambda)(z) = \frac{f(z) - f(\lambda)}{z - \lambda}$ for $z \neq \lambda$ and $\Phi(f, \lambda)(\lambda) = f'(\lambda)$. Then $\Phi(f, \lambda) \in A^+$ and, for $\lambda \in D$,

$$\|\Phi(f, \lambda)\|_{A^+} \leq \frac{2\|f\|_{A^+}}{1 - |\lambda|}.$$

If, in addition, $f \in I^+(E)$, then

$$f(\lambda)(\pi(\alpha) - \lambda)^{-1} = -\pi(\Phi(f, \lambda)),$$

where $\pi$ is the canonical map from $A^+$ onto $A^+/I^+(E)$.
Proof. Set Lemma 1 and [2, Example 1.3]. Recall that
\[ \|e^{int}\|_{A^+(E)} = \|\pi(\alpha)^{n}\|_{A^+/I^+(E)}, \quad n \in \mathbb{Z} \]
where \( \alpha : e^{i\theta} \rightarrow e^{i\theta} \) for \( \theta \in [0, 2\pi] \).
If \( f(\lambda) \neq 0 \), we obtain from (2):
\[ (\pi(\alpha) - \lambda)^{-1} \|_{A^+/I^+(E)} \leq \frac{1}{|f(\lambda)|} \frac{2\|f\|_{A^+}}{|1 - |\lambda||}. \]

Theorem 3. If \( E \) satisfies (1), for some \( p \in \mathbb{R} \), then \( E \) is a \( ZA^+ \) set and there exists \( c > 0 \) such that
\[ \omega_n = 0(\exp cn^{1/(1+p)}). \]
Proof. As in [3], we define \( h \) by
\[ h(t) = K \left( \log \frac{2\pi}{t - \alpha_\nu} + \log \frac{2\pi}{\beta_\nu - t} \right), \quad t \in [\alpha_\nu, \beta_\nu], \quad \nu \in \mathbb{N}. \]
Using Proposition 1, we see that \( h \in L^p(\Gamma) \subset L^1(\Gamma) \), so the function
\[ f(z) = \exp \left( -\frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{it} + z}{e^{it} - z} h(t) \, dt \right) \]
is analytic in \( D \).
We have \( f \in A^+ \) for \( K > 2 \) [3], and
\[ |f(e^{it})| = \lim_{r \to 1} |f(re^{it})| = \left\{ \frac{(t - \alpha_\nu)(\beta_\nu - t)}{4\pi^2} \right\}^K. \]
Thus \( f|_E = 0 \), and \( E \) is a \( ZA^+ \) set.
Let
\[ g(z) = \frac{1}{f(z)} = \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{it} + z}{e^{it} - z} h(t) \, dt \right) \quad (z \in D). \]
We have
\[ \log |g(re^{i\theta})| = \frac{1}{2\pi} \int_{0}^{2\pi} P_r(\theta - t) h(t) \, dt \quad (0 \leq r < 1). \]
Since \( P_r(\theta - t) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in(\theta - t)} \), we obtain
\[ \log |g(re^{i\theta})| = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \sum_{n \in \mathbb{Z}} r^{|n|} e^{in(\theta - t)} h(t) \right) \, dt \]
\[ = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} r^{|n|} \left( \int_{0}^{2\pi} e^{-in t} h(t) \, dt \right) e^{in\theta} \]
\[ = \sum_{n \in \mathbb{Z}} r^{|n|} \hat{h}(n) e^{in\theta}. \]
Recall that if \( h \in L^p \) for \( 1 < p \leq 2 \), then \( \hat{h}(n) \in l^q(\mathbb{Z}) \) and \( \{\sum_{n \in \mathbb{Z}} |\hat{h}(n)|^q\}^{1/q} \leq \|h\|_p \) where \( \frac{1}{p} + \frac{1}{q} = 1 \) [7, p. 98].

So we have
\[
\log |g(re^{i\theta})| \leq \left\{ \sum_{n \in \mathbb{Z}} (r|n|)^p \right\}^{1/p} \|h\|_p \\
\leq 2^{1/p} \frac{1}{(1-r)^{1/p}} \|h\|_p,
\]
and by (2) we have
\[
\|((\pi(\alpha) - \lambda)^{-1})_{A^*\Gamma} \leq \frac{2\|f\|_{A^*} \exp 2^{1/p} \|h\|_p (1 - |\lambda|)^{-1/p}}{1 - |\lambda|}.
\]

Using Lemma 2 in [2] we obtain that there exists \( b, c > 0 \) such that
\[
\|\pi(\alpha)^{-n}\| \leq b \exp c n^{1/(1+p)} \quad \text{for } n \geq 0.
\]

This proves the theorem.

We give an application of this method to some other closed ideals.

Let \( I \) be a closed ideal of \( A^* \), and let
\[
h(I) = \{z \in \overline{D} \text{ such that } f(z) = 0 \text{ for every } f \in I\}.
\]

A consequence of Taylor's and Williams's estimates [8, Lemmas 5.8 and 5.9] is that, if \( f \in A^* \) and \( f(e^{i\theta}) = 0(\text{dist}(e^{i\theta}, h(I))^2) \), and if the only common inner factor of elements of \( I \) is 1, then \( f \in I \) [4].

Using this result we obtain:

**Theorem 4.** Let \( I \) be a closed ideal of \( A^* \) and let \( \pi: A^* \to A^*/I \) be the canonical map. If \( I \) is such that:

(a) \( h(I) \) satisfies (1) for some \( p \in ]1, 2[ \), and

(b) the only common inner factor of all elements of \( I \) is 1,

then there exists \( c > 0 \) such that
\[
\|\pi(\alpha)^{-n}\|_{A^*/I} = 0(\exp c n^{1/(1+p)})
\]

**Proof.** From (b) we have \( h(I) \subset \Gamma \). Since \( h(I) \) satisfies (1), if \( f \) is the function defined in theorem 3 for \( E = h(I) \), then \( f \) verifies
\[
f(e^{i\theta}) = 0(\text{dist}(e^{i\theta}, h(I)))^{K}, \quad K > 2.
\]

Thus \( f \in I \).

We see as above that
\[
\|((\pi(\alpha) - \lambda)^{-1}) \| \leq \frac{2\|f\|_{A^*}}{1 - |\lambda|},
\]
and the estimates for \( \|\pi(\alpha)^{-n}\| \) follow from the same argument as in the proof of Theorem 3.
References


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