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SPREAD AND LOCAL PROPERTIES

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ABSTRACT. A technique for combining the spread of a space and several local properties is exploited to obtain decompositions of a space and cardinal function bounds on the size and weight. It is shown that a locally countable set is the union of ω_1 discrete sets and that under $MA(\omega_1)$ the weight of a manifold is equal to its spread.

1. INTRODUCTION

The main technique used in this paper is taken from a lemma the author used to prove that $s[C_p(X)^\omega] \leq s[C_p(X)]^+$ ($C_p(X) =$ space of continuous real-valued functions with the topology of pointwise convergence) [3]. It is a variant of a well-known (and useful) fact about CCC spaces: if $\mathscr{U}$ is an open collection, then there is a countable subcollection $\mathscr{V}$ such that the union of $\mathscr{V}$ is dense in the union of $\mathscr{U}$ [4], [6]. This fact is easily proven by considering a maximal cellular open refinement. The method used here is to repeatedly take maximal cellular subcollections and then to use the knowledge of a local property to show when this process stops. This avoids using closure to get a cover of the original open collection. The applications often yield bounds in terms of a successor cardinal, as in the inequality for $C_p(X)$ above. Not surprisingly, the sharpness of the bounds depends on extra set-theoretic assumptions.

We start by considering subsets of a space which are locally $< \lambda$ for some infinite cardinal λ (each point has a relatively open neighborhood of cardinality $< \lambda$).

1.1. **Theorem.** *If $A \subseteq X$ is locally $< \lambda$, then A is the union of λ discrete subsets.*

Proof. The proof may be achieved with relatively open sets, so without loss of generality assume that $A = X$. Let $\mathscr{V} = \{V_x : x \in X\}$ be a collection of open

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neighborhoods such that, for each $x \in X$,

(1) $x \in V_x$ and
(2) $|V_x| < \lambda$.

For $\alpha \in \lambda$ inductively define sets A_α and S_α such that

(a) $S_\alpha \subseteq A_\alpha \subseteq X \setminus \bigcup_{\beta\in\alpha} A_\beta$;
(b) S_α is discrete;
(c) if $X \setminus (\bigcup_{\beta\in\alpha} A_\beta)$ is nonempty, then $\{V_x \setminus (\bigcup_{\beta\in\alpha} A_\beta) : x \in S_\alpha\}$ is a maximal pairwise disjoint collection in $X \setminus (\bigcup_{\beta\in\alpha} A_\beta)$, and $A_\alpha = \bigcup\{V_x : x \in S_\alpha\} \setminus (\bigcup_{\beta\in\alpha} A_\beta)$.

If $X \setminus (\bigcup_{\beta\in\alpha} A_\beta)$ is empty, then $S_\alpha = A_\alpha = \emptyset$.

Claim. $\bigcup\{A_\alpha : \alpha \in \lambda\} = X$.

Assume the contrary, and let $x \in X \setminus (\bigcup_{\beta\in\lambda} A_\beta)$. Then, by the maximality in (c), we have that V_x meets A_α for each $\alpha \in \lambda$. However, condition (a) causes the A_α's to be pairwise disjoint. Hence the cardinality of V_x must be at least λ, which is a contradiction.

Now we can define the discrete subspaces. Fix $\alpha \in \lambda$. A_α is covered by the pairwise disjoint, relatively open collection $\{V_x \setminus (\bigcup_{\beta\in\alpha} A_\beta) : x \in S_\alpha\}$. Furthermore, each $V_x \setminus (\bigcup_{\beta\in\alpha} A_\beta)$ can be written as $\{z_{x,\beta} : \beta \in \lambda\}$ (repeat an element). So, for each $\beta \in \lambda$, let $D_{\alpha,\beta} = \{z_{x,\beta} : x \in S_\alpha\}$. Then for each $y \in S_\alpha$, $D_{\alpha,\beta} \cap V_y = \{z_{y,\beta}\}$. Hence $D_{\alpha,\beta}$ is discrete. So, by the claim, $X = \bigcup\{D_{\alpha,\beta} : \alpha, \beta \in \lambda\}$ is the union of λ discrete sets. □

Theorem 1.1 also has the following set-theoretic version:

1.2. **Theorem.** *Let $\mathscr{V} \subseteq [E]^{<\lambda}$ be a cover of a set E. Then there is a partition $\{A_\alpha : \alpha \in \lambda\}$ of E and a sequence of subcollections of $\mathscr{V}$, $\{\mathscr{V}_\alpha : \alpha \in \lambda\}$, such that $\{V \cap A_\alpha : V \in \mathscr{V}_\alpha\}$ is a partition of A_α.*

1.3. **Corollary.** *If X is locally $< \lambda$, then $|X| \leq \lambda \cdot s(X)$. (Hence locally countable S-spaces have cardinality at most ω_1.)*

The study of S-spaces often begins by reducing the problem to a subset of 2^{ω_1}. By standard techniques, if X is an S-space, then X contains a locally countable S-subspace. Consequently under CH it contains a locally countable S-subspace of weight at most ω_1.

1.4. **Corollary.** *If X is a locally countable Tychonoff space with countable spread and $\chi(X) \leq \omega_1$, then $w(X) \leq \omega_1$, and hence X can be embedded in 2^{ω_1}. (Consequently all the locally compact* [*or* 1st *countable*]*, locally countable S-spaces can be embedded in 2^{ω_1}.)*

Proof. $w(X) \leq |X| \cdot \chi(X)$.

As a comparison to Theorem 1.1, consider the following theorem of Balogh:

1.5. **Theorem** (Balogh [2]). $[MA + \neg CH]$ *In a compact space X of countable tightness, every locally countable subset of cardinality $< 2^\omega$ is the countable union of discrete sets.*

2. Results using hereditary Lindelöf degree

In this section, the cardinality condition in Theorem 1.1 is replaced with the hereditary Lindelöf degree with similar results. In fact, under CH we find that local countability can be weakened to locally hereditarily Lindelöf in Theorem 1.1.

2.1. **Theorem.** *If a T_2 space X is locally hereditarily λ-Lindelöf, then X has an open cover $\mathscr{V} = \bigcup\{\mathscr{V}_\alpha : \alpha \in \lambda^+\}$ of hereditarily λ-Lindelöf sets such that $\forall \alpha \in \lambda^+$, $\mathscr{V}_\alpha$ witnesses a discrete set S_α. Also, X can be written as the union of 2^λ discrete sets.*

Proof. Fix an open cover $\mathscr{W} = \{V_x : x \in X\}$ such that for each $x \in X$, $x \in V_x$ and $hL(V_x) \le \lambda$. For $\alpha \in \lambda^+$, inductively define the sets A_α and S_α such that

(a) $S_\alpha \subseteq A_\alpha \subseteq X \setminus \bigcup_{\beta\in\alpha} A_\beta$;
(b) S_α is discrete;
(c) if $X \setminus (\bigcup_{\beta\in\alpha} A_\beta)$ is nonempty, then $\{V_x \setminus (\bigcup_{\beta\in\alpha} A_\beta) : x \in S_\alpha\}$ is a maximal pairwise disjoint collection in $X \setminus (\bigcup_{\beta\in\alpha} A_\beta)$ and $A_\alpha = \bigcup\{V_x : x \in S_\alpha\} \setminus (\bigcup_{\beta\in\alpha} A_\beta)$.

If $X \setminus (\bigcup_{\beta\in\alpha} A_\beta)$ is empty, then $S_\alpha = A_\alpha = \varnothing$.

Claim. $X = \bigcup\{A_\alpha : \alpha \in \lambda^+\}$, and hence X is covered by $\bigcup\{\mathscr{V}_\alpha : \alpha \in \lambda^+\}$, where $\mathscr{V}_\alpha = \{V_x : x \in S_\alpha\}$.

Assume the contrary, and let $x \in X \setminus \bigcup\{A_\alpha : \alpha \in \lambda^+\}$. Consider the following collection:

$$(*) \qquad \left\{ \left(\bigcup_{\beta\in\alpha} A_\beta \right) \cap V_x : \alpha \in \lambda^+ \right\}.$$

As in Theorem 1.1, $V_x \cap A_\alpha \neq \varnothing$ for each $\alpha \in \lambda^+$ by the maximality in (c). Hence the above collection $(*)$ is strictly increasing. Also note that by the definition of the A_β's for each $\alpha \in \lambda^+$,

$$\bigcup_{\beta\in\alpha} A_\beta = \bigcup \left\{ V_y : y \in \bigcup_{\beta\in\alpha} S_\beta \right\},$$

and hence is open. Therefore $(*)$ is a strictly increasing sequence of open sets in V_x contradicting the assumption that $hL(V_x) \le \lambda$.

Recall that for any T_2 space Y, $|Y| \le 2^{hL(Y)}$ (see Hodel [6]). So for each $\alpha \in \lambda^+$ and $x \in S_\alpha$ write each $V_x \setminus (\bigcup_{\beta\in\alpha} A_\beta)$ as $\{z_{x,\gamma} : \gamma \in 2^\lambda\}$ (repeat an element if necessary). Then, as in Theorem 1.1, the set $D_{\alpha,\gamma} = \{z_{x,\gamma} : x \in S_\alpha\}$ is discrete and $X = \bigcup\{D_{\alpha,\gamma} : \alpha \in \lambda^+ \wedge \gamma \in 2^\lambda\}$. □

2.2. **Theorem.** *If X is a T_2 space with an open cover $\mathscr{V}$ of sets with hereditary Lindelöf degree at most λ, then $hL(X) \leq \lambda^+ \cdot s(X)$ and $|X| \leq 2^\lambda \cdot s(X)$.*

2.3. **Theorem.** *Let $\lambda < \kappa$ be infinite cardinals. If each point x in a T_2 space X has a neighborhood V_x such that $hL(V_x) \leq \lambda$ and $w(V_x) \leq \kappa$, then $w(X) \leq \kappa \cdot s(X)$.*

Proof. Let $\mathscr{V} = \{V_\alpha : \alpha \in \lambda^+ \cdot s(X)\}$ be an open cover of sets of weight at most κ. Fix a base $\mathscr{B}_\alpha$ of cardinality $\leq \kappa$ for each $V_\alpha \in \mathscr{V}$. Then, since each V_α is open and $X = \bigcup \mathscr{V}$, $\mathscr{B} = \bigcup\{\mathscr{B}_\alpha : \alpha \in \lambda^+ \cdot s(X)\}$ is a base for the entire space of cardinality, at most $\lambda^+ \cdot \kappa \cdot s(X) = \kappa \cdot s(X)$. □

2.4. **Corollary.** *If X is locally compact, then $w(X) \leq \Delta(X)^+ \cdot s(X)$ and $|X| \leq 2^{\Delta(X)} \cdot s(X)$. In particular, if X has a G_δ-diagonal, then $w(X) \leq \omega_1 \cdot s(X)$ and $|X| \leq 2^\omega \cdot s(X)$.*

Proof. For a compact neighborhood V_x, we have $hL(V_x) \leq w(V_x) \leq \Delta(X)$. So let $\lambda = \Delta(X)$ and $\kappa = \Delta(X)^+$ in Theorems 2.2 and 2.3. □

2.5. **Corollary.** *For any locally metrizable space, $w(X) \leq s(X)^+$ and $|X| \leq 2^{s(X)}$.*

Proof. For a metrizable V, $hL(V) = w(V) \leq s(X)$. So let $\lambda = s(X)$ and $\kappa = s(X)^+$ in Theorems 2.2 and 2.3. □

2.6. **Corollary.** *If X is locally second countable such that $s(X) \neq w(X)$, then $s(X) = \omega$ and $w(X) = \omega_1$.*

It is known that $hd(X) = s(X) \leq L(X) = w(X)$ for any manifold X. In addition, M. E. Rudin has constructed a manifold under $\diamondsuit$ with countable spread and uncountable weight. (See [8].) Hence Corollaries 2.5 and 2.6 are the best possible ZFC statements. Under $MA(\omega_1)$, however, for any manifold we have equality for each of the four cardinals.

2.7. **Theorem** $[MA(\omega_1)]$. *If a space X is both locally compact and locally metrizable, then $s(X) = w(X)$ (i.e., it is consistent that $s(X) = w(X)$ for all manifolds).*

Proof. Use the fact that under $MA(\omega_1)$ there are no locally compact S-spaces. (See Balogh's paper [2].) □

2.8. **Theorem.** *If there is a space X of countable spread such that X is either*

(a) *locally compact with a G_δ-diagonal and $w(X) = 2^\omega$; or*
(b) *locally metrizable (second countable) with $w(X) = 2^\omega$; or*
(c) *locally countable with $|X| = 2^\omega$,*

then the Continuum Hypothesis holds.

2.9. **Example** (Kunen [7]). (CH) The Kunen line is a locally countable, locally compact, locally metrizable space X with countable spread, a G_δ-diagonal, and $w(X) = |X| = 2^\omega$.

As an alternate view of the above, consider the fact that it is consistent with $MA + \neg CH$ that there is a first countable S-space but there are no first countable L-spaces (see [1] and [9]). In both cases we have $s(X) = \omega$; hence, the following applies equally to both situations.

2.10. **Theorem.** *If $s(X) = \omega$ and $w(X) \geq \omega_2$ (or $w(X) \geq 2^\omega$ under $\neg CH$), then X contains a point with no second countable neighborhood.*

We conclude this section with several results of Balogh which use hereditary normality and Jones' lemma to provide a bound on the spread. For these we will need extra set-theoretic assumptions. Let $(\mathscr{E})$ be the following statement about the exponential function implied by both MA and CH:

$$(\mathscr{E}) \qquad \text{for each cardinal } \kappa,\ \kappa < 2^\omega \Rightarrow 2^\kappa < 2^{2^\omega}.$$

2.11. **Proposition** (Jones' Lemma). *Assume $(\mathscr{E})$. For any T_5 space X, if $d(X) < 2^\omega$, then X has no discrete sets of size 2^ω.* □

2.12. **Theorem** (Balogh [2]). *Assume $(\mathscr{E})$. If $cf(2^\omega) > \omega_1$, then every locally hereditarily Lindelöf, T_5 of density $< 2^\omega$ has Lindelöf degree $< 2^\omega$.*

Proof. For each $x \in X$, fix a hereditarily Lindelöf neighborhood V_x. By Theorem 2.1, there is a subcollection, $\mathscr{W}$, of these neighborhoods such that

(1) $\mathscr{W} = \bigcup\{\mathscr{W}_\alpha : \alpha \in \omega_1\}$ is a cover of X and
(2) each $\mathscr{W}_\alpha$ witnesses a discrete set.

By Jones' Lemma, $|\mathscr{W}_\alpha| < 2^\omega$ for each α, and by the cofinality restriction, $|\mathscr{W}| < 2^\omega$. □

2.13. **Theorem** (Balogh [2]). *Assume $(\mathscr{E})$. If $cf(2^\omega) > \omega_1$, then every point in a locally hereditarily Lindelöf, locally separable, T_5 space is contained in a clopen set of Lindelöf degree $< 2^\omega$.*

Proof. For each $x \in X$ construct, using Theorem 2.12, an increasing sequence of open sets $\{W_\alpha : \alpha \in \omega_1\}$ containing x such that for each $\alpha \in \omega_1$

(a) $\overline{W_\alpha} \subseteq W_{\alpha+1}$ and
(b) $d(W_\alpha) < 2^\omega$ (local separability and Theorem 2.12).

Then the union will be clopen by the local separability and will have Lindelöf degree $< 2^\omega$ by Theorem 2.12. □

2.14. *Remarks.* (i) Balogh assumes the space is locally hereditarily separable, but needs to consider only the density of open sets. See his paper [2] and Fremlin's [5] for more discussion on the above theorem and the consequences of combining MA or PFA with local properties.

(ii) In Theorems 1.1 and 2.1, we can fix the open cover $\mathscr{V}$ that we are working with first, and then replace the spread of X with a bound on the size of a disjoint subcollection of $\{V \cap A : V \in \mathscr{V}\}$, where $A \subseteq \bigcup \mathscr{V}$. This was necessary to prove that $s[C_p(X)^\omega] \leq s[C_p(X)]^+$ [3].

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