PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 3, July 1991

AUSLANDER-REITEN TRIANGLES IN DERIVED CATEGORIES OF FINITE-DIMENSIONAL ALGEBRAS

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(Communicated by Maurice Auslander)

ABSTRACT. Let A be a finite-dimensional algebra. The category $\operatorname{mod} A$ of finitely generated left A-modules canonically embeds into the derived category $D^b(A)$ of bounded complexes over $\operatorname{mod} A$ and the stable category $\underline{\operatorname{mod}}^{\mathbb{Z}}\, T(A)$ of $\mathbb{Z}$-graded modules over the trivial extension algebra of A by the minimal injective cogenerator. This embedding can be extended to a full and faithful functor from $D^b(A)$ to $\underline{\operatorname{mod}}^{\mathbb{Z}}\, T(A)$. Using the concept of Auslander-Reiten triangles it is shown that both categories are equivalent only if A has finite global dimension.

Let k be a field and A a finite-dimensional k-algebra. By $\operatorname{mod} A$ we denote the category of finitely generated left A-modules. Let $T(A)$ be the trivial extension algebra by the bimodule $\operatorname{Hom}_k(A, k)$. Then $T(A)$ is a $\mathbb{Z}$-graded algebra and the category $\operatorname{mod}^{\mathbb{Z}} T(A)$ of finitely generated $\mathbb{Z}$-graded $T(A)$-modules is a Frobenius category in the sense of [H1]. The stable category of $\operatorname{mod}^{\mathbb{Z}} T(A)$ is denoted by $\underline{\operatorname{mod}}^{\mathbb{Z}}\, T(A)$. In [H1] (see also [H2, KV]) we show that the derived category $D^b(A) = D^b(\operatorname{mod} A)$ and $\underline{\operatorname{mod}}^{\mathbb{Z}}\, T(A)$ are triangle-equivalent, if the global dimension of A is finite. The purpose of this note is to prove the converse of this result. We point out that examples of this fact have been obtained in [TW]. Moreover we will give a surprisingly simple construction of a full and faithful exact functor F from $D^b(A)$ to $\underline{\operatorname{mod}}^{\mathbb{Z}}\, T(A)$.

The proof is based on the fact that $\underline{\operatorname{mod}}^{\mathbb{Z}}\, T(A)$ has Auslander-Reiten triangles in the sense of [H1], whereas we will show in section one that $D^b(A)$ has Auslander-Reiten triangles only if A is of finite global dimension. Thus obtaining the converse of a result established in [H1].

In §2 we will give the construction of F and the proof of the aforementioned theorem.

The composition of two morphisms $f: X \to Y$ and $g: Y \to Z$ in a given category is denoted by fg.

Received by the editors March 23, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 16A46, 16A64, 18E30.
Key words and phrases. Repetitive algebras, Auslander-Reiten triangles.

1. AUSLANDER-REITEN TRIANGLES IN $D^b(A)$

In [H1] we introduced the notion of an Auslander-Reiten triangle in a triangulated category. We first recall the relevant definitions.

1.1. Let $\mathscr{C}$ be a triangulated category such that $\operatorname{Hom}_{\mathscr{C}}(X, Y)$ is a finite-dimensional k-vector space for all $X, Y \in \mathscr{C}$ and assume that the endomorphism ring of an indecomposable object is local. This assumption ensures that $\mathscr{C}$ is a Krull-Schmidt category (compare 2.2 of [R]). We denote by $X[1]$ the value of the translation functor on the object X of $\mathscr{C}$.

A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ in $\mathscr{C}$ is called an Auslander-Reiten triangle if the following conditions are satisfied:

(AR1) X, Z are indecomposable;
(AR2) $w \neq 0$;
(AR3) if $f : W \to Z$ is not a retraction, then there exists $f' : W \to Y$ such that $f'v = f$.

We will say that $\mathscr{C}$ has Auslander-Reiten triangles if for all indecomposable objects $Z \in \mathscr{C}$ there exists a triangle satisfying the conditions above.

1.2. The following observations were obtained in [H1]. For more details and related results we refer to [H1, H2].

First note that the following are equivalent for a triangle as above:

(i) (AR2);
(ii) u is not a section;
(iii) v is not a retraction.

If this is not the case we infer that $Y \simeq X \oplus Z$. Indeed, let $w = 0$. Then we obtain the following diagram of triangles in $\mathscr{C}$

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{0} & X[1] \\ {\scriptstyle 1_X}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle 1_Z}\downarrow & & {\scriptstyle 1_{X[1]}}\downarrow \\ X & \xrightarrow{\mu} & X \oplus Z & \xrightarrow{\pi} & Z & \xrightarrow{0} & X[1] \end{array}$$

where μ and π denote the canonical maps. Note that the second row is indeed a triangle, since it follows from the octahedral axiom that the direct sum of triangles is a triangle. Now we obtain by the third axiom of a triangulated category (see for example [V]) a morphism $f : Y \to X \oplus Z$, which is an isomorphism, by a well-known fact in triangulated categories (see for example [V]).

And also the following are equivalent for a triangle as above:

(i) (AR3);
(ii) If $f : W \to Z$ is not a retraction, then $fw = 0$.

1.3. In the proof of the following theorem we will need some facts about $D^b(A)$ as well as the following notation.

Let $\mathfrak{a}$ be an arbitrary additive subcategory of $\operatorname{mod} A$. A complex $X^\bullet = (X^i, d_X^i)_{i\in\mathbb{Z}}$ over $\mathfrak{a}$ is a collection of objects X^i and morphisms $d^i = d_X^i : X^i \to X^{i+1}$ such that $d^i d^{i+1} = 0$. A complex $X^\bullet = (X^i, d_X^i)$ is bounded below if $X^i = 0$ for all but finitely many $i < 0$. It is called bounded above if $X^i = 0$ for all but finitely many $i > 0$. It is bounded if it is bounded below and bounded above. It is said to have bounded cohomology if $H^i(X^\bullet) = 0$ for all but finitely many $i \in \mathbb{Z}$, where by definition $H^i(X^\bullet) = \ker d_X^i / \operatorname{im} d_X^{i-1}$. Denote by $C(\mathfrak{a})$ the category of complexes over $\mathfrak{a}$, by $C^{-,b}(\mathfrak{a})$ (resp. $C^{+,b}(\mathfrak{a})$, resp. $C^b(\mathfrak{a})$) the full subcategories of complexes bounded above with bounded cohomology (resp. bounded below with bounded cohomology, resp. bounded above and below).

If $X^\bullet = (X^i, d_X^i)_{i\in\mathbb{Z}}$ and $Y^\bullet = (Y^i, d_Y^i)_{i\in\mathbb{Z}}$ are two complexes, a morphism $f^\bullet : X^\bullet \to Y^\bullet$ is a sequence of morphisms $f^i : X^i \to Y^i$ of $\mathfrak{a}$ such that

$$d_X^i f^{i+1} = f^i d_Y^i$$

for all $i \in \mathbb{Z}$.

The translation functor is defined by $(X^\bullet[1])^i = X^{i+1}$, $(d_{X[1]})^i = -(d_X)^{i+1}$.

The mapping cone $C_{f^\bullet}$ of a morphism $f^\bullet : X^\bullet \to Y^\bullet$ is the complex

$$C_{f^\bullet} = ((X^\bullet[1])^i \oplus Y^i, d_{C_f}^i)$$

with 'differential'

$$d_{C_f}^i = \begin{pmatrix} -d_X^{i+1} & f^{i+1} \\ 0 & d_Y^i \end{pmatrix}.$$

We denote by $K^{-,b}(\mathfrak{a})$, $K^{+,b}(\mathfrak{a})$, and $K^b(\mathfrak{a})$ the homotopy categories of the categories of complexes introduced above.

Recall that two morphisms $f^\bullet, g^\bullet : X^\bullet \to Y^\bullet$ are called homotopic if there exist morphisms $h^i : X^i \to Y^{i-1}$ such that $f^i - g^i = d_X^i h^{i+1} + h^i d_Y^{i-1}$ for all $i \in \mathbb{Z}$.

For instance if $Z^\bullet \in C(\mathfrak{a})$ satisfies $Z^i = 0$ for $i > 0$, and $Z_n^\bullet$ for $n < 0$ is the truncated complex with $Z_n^i = 0$ for $i < n$, $Z_n^i = Z^i$ for $i \geq n$, $d_{Z_n}^i = d_Z^i$ for $i \geq n$, and zero otherwise, we obtain a morphism $\mu_n^\bullet : Z_n^\bullet \to Z^\bullet$ with $\mu_n^i = id_{Z^i}$ for $i \geq n$ and zero otherwise whose mapping cone is isomorphic in the homotopy category to $Z_n'^\bullet$ with $Z_n'^i = 0$ for $i \geq n$ and $Z_n'^i = Z^i$ for $i < n$ and $d_{Z_n'}^i = d_Z^i$ for $i < n-1$ and zero otherwise. We denote by $\pi_n^\bullet$ the induced morphism from $Z^\bullet$ to $Z_n'^\bullet$. Thus $\pi_n^i = id_{Z_n}$ for $i < n$ and zero otherwise. Note that d^{n-1} induces a morphism $d_n^\bullet$ from $Z_n'^\bullet$ to $Z_n^\bullet[1]$. In particular we obtain a triangle in the homotopy category

$$Z_n^\bullet \xrightarrow{\mu_n^\bullet} Z^\bullet \xrightarrow{\pi_n^\bullet} Z_n'^\bullet \xrightarrow{d_n^\bullet} Z_n^\bullet[1].$$

This construction will turn out to be quite useful in the next subsection.

We denote by ${}_A\mathscr{P}$ (resp. ${}_A\mathscr{I}$) the full subcategory of $\operatorname{mod} A$ formed by the projective (resp. injective) A-modules. Then we identify the derived category $D^b(A)$ of bounded complexes over $\operatorname{mod} A$ with $K^{-,b}({}_A\mathscr{P})$ or with $K^{+,b}({}_A\mathscr{I})$. In case A has finite global dimension this yields the identification of $D^b(A)$ with $K^b({}_A\mathscr{P})$ or with $K^b({}_A\mathscr{I})$, since the natural embedding of $K^b({}_A\mathscr{P})$ into $K^{-,b}({}_A\mathscr{P})$ is an equivalence in this case.

1.4. In [H1] we proved that $D^b(A)$ has Auslander-Reiten triangles if the global dimension of A is finite. Here we present now the following generaliztion.

Theorem. *Let A be a finite-dimensional k-algebra. Then*

(i) *Let $Z^\bullet \in K^{-,b}({}_A\mathscr{P})$ be indecomposable. Then there exists an Auslander-Reiten triangle $X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$ if and only if $Z^\bullet \in K^b({}_A\mathscr{P})$.*

(ii) *Let $X^\bullet \in K^{+,b}({}_A\mathscr{I})$ be indecomposable. Then there exists an Auslander-Reiten triangle $X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$ if and only if $X^\bullet \in K^b({}_A\mathscr{I})$.*

Proof. We will prove (i). The second assertion follows by duality.

If $Z^\bullet \in K^b({}_A\mathscr{P})$ is indecomposable the construction of the Auslander-Reiten triangle

$$X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$$

is identical as in the proof of 3.6 of [H1]. For the convenience of the reader we sketch the argument.

First observe that ${}_A\mathscr{P}$ and ${}_A\mathscr{I}$ are equivalent under the Nakayama functor $\nu = D\operatorname{Hom}_A(-, {}_AA)$, where D denotes the duality on $\operatorname{mod} A$ with respect to the base field k. There is also an invertible natural transformation $\alpha_P : D\operatorname{Hom}(P, -) \to \operatorname{Hom}(-, \nu P)$ for $P \in \mathscr{P}$.

This induces an equivalence of triangulated categories again denoted by ν between $K^b({}_A\mathscr{P})$ and $K^b({}_A\mathscr{I})$ and an invertible natural transformation $\alpha_{P^\bullet} : D\operatorname{Hom}(P^\bullet, -) \to \operatorname{Hom}(-, \nu P^\bullet)$ for $P^\bullet \in K^b({}_A\mathscr{P})$.

Now assume $Z^\bullet$ is indecomposable in $K^b({}_A\mathscr{P})$. Let φ in $D\operatorname{Hom}(Z^\bullet, Z^\bullet)$ be a linear form on $\operatorname{End} Z^\bullet$ which vanishes on the radical $\operatorname{rad}\operatorname{End} Z^\bullet$ and satisfies $\varphi(id_{Z^\bullet}) = 1$. We consider the image $\alpha_{Z^\bullet}(\varphi)$; it is a nonzero linear map from $Z^\bullet$ to $\nu Z^\bullet$ such that $f\alpha_{Z^\bullet}(\varphi) = 0$ whenever the morphism f of $D^b(A)$ is not a retraction. Consider the triangle

$$\nu Z^\bullet[-1] \to Y^\bullet \to Z^\bullet \xrightarrow{\alpha_{Z^\bullet}(\varphi)} \nu Z^\bullet$$

having $\alpha_{Z^\bullet}(\varphi)$ as last morphism. We infer that this is an Auslander-Reiten triangle.

We are thankful to the referee for pointing out that the converse direction follows by applying proposition 3.2 of [Ri1]. For the convenience of the reader we have included the elementary direct proof.

For the converse let $Z^\bullet \in K^{-,b}({}_A\mathcal{P})$ be indecomposable. Applying the translation functor if necessary, we may assume that $Z^i = 0$ for $i > 0$. Assume that $Z^\bullet$ is not isomorphic to some complex in $K^b({}_A\mathcal{P})$ and let

$$X^\bullet \xrightarrow{u^\bullet} Y^\bullet \xrightarrow{v^\bullet} Z^\bullet \xrightarrow{w^\bullet} X^\bullet[1]$$

be the Auslander-Reiten triangle. Set $W^\bullet = X^\bullet[1]$.

Let $n < 0$ and consider the morphism $\mu_n^\bullet : Z_n^\bullet \to Z^\bullet$. We claim that $\mu_n^\bullet$ is not a retraction for all $n < 0$. Assume that $\mu_n^\bullet$ is a retraction. Then $Z_n^\bullet \simeq Z_n'^\bullet[-1] \oplus Z^\bullet$ by the remark in 1.2. Since $Z^\bullet$ is not isomorphic to a complex in $K^b({}_A\mathcal{P})$ we obtain a contradiction.

From 1.2 it follows that $\mu_n^\bullet w^\bullet = 0$ in $K^{-,b}({}_A\mathcal{P})$, (i.e. homotopic to zero) for all $n < 0$. Let $h^i : Z_n^i \to W^{i-1}$ such that $\mu_n^i w^i = d_{Z_n} h^{i+1} + h^i d_W^{i-1}$ for all $i \in \mathbb{Z}$. Since $W^\bullet \in K^{-,b}({}_A\mathcal{P})$ there is $m_0 < 0$ such that $H^m(W^\bullet) = 0$ for all $m \le m_0$.

Let $n \le m^0$.

We now define inductively $g^i : Z^i \to W^{i-1}$ such that $w^i = d_Z^i g^{i+1} + g^i d_W^{i-1}$ in contradiction to (AR2).

For $i > m_0$ we let $g^i = h^i$. By the considerations above and the fact that $\mu_n^i = id_{Z^i}$ for $i \ge n$ we infer that

$$w^i = d_Z g^{i+1} + g^i d_W^{i-1} \quad \text{for } i > m_0.$$

Now suppose that $g^i : Z^i \to W^{i-1}$ is defined such that $w^i = d_Z g^{i+1} + g^i d_W^{i-1}$ for $i > r \ge m_0$. Let $\pi^{r-1}\mu^{r-1} = d_W^{r-1}$ be the canonical factorisation of d_W^{r-1}. Consider $w^r - d_Z^r g^{r+1} : Z^r \to W^r$. Then

$$\begin{aligned}(w^r - d_Z^r g^{r+1})d_W^r &= w^r d_W^r - d_Z^r g^{r+1} d_W^r \\ &= w^r d_W^r - d_Z^r(w^{r+1} - d_Z^{r+1} g^{r+2}) \\ &= w^r d_W^r - d_Z^r w^{r+1} \\ &= 0.\end{aligned}$$

Thus there exists $g' : Z^r \to \ker d_W^r$ such that $g'\mu^{r-1} = w^r - d_Z^r g^{r+1}$. Since Z^r is a projective A-module there exists $g^r : Z^r \to W^{r-1}$ such that $g^r \pi^{r-1} = g'$. Now

$$\begin{aligned}d_Z^r g^{r+1} + g^r d_W^{r-1} &= d_Z^r g^{r+1} + g^r \pi^{r-1} \mu^{r-1} \\ &= d_Z^r g^{r+1} + g' \mu^{r-1} \\ &= w^r\end{aligned}$$

completes the induction.

1.5. The following corollary is an immediate consequence of the theorem.

Corollary. *Let A be a finite-dimensional k-algebra. Then $D^b(A)$ has Auslander-Reiten triangles if and only if A has finite global dimension.*

2. Repetitive algebras

For the reader's convenience we recall relevant definitions, but refer to [H1, H2] for more detail and proofs of some elementary facts.

2.1. Let A be a finite-dimensional k-algebra, where k is some field. Denote by $D = \mathrm{Hom}_k(-, k)$ the standard duality on $\mathrm{mod}\, A$ the category of finitely generated left A-modules. Let $Q = DA$. Then Q is an injective cogenerator and an A-bimodule: given $a, b \in A$ and $\varphi \in Q$ then $a\varphi b$ is the k-linear map which sends $\lambda \in A$ to $\varphi(b\lambda a)$.

Let us construct the repetitive algebra $\hat{A}$. It will be a selfinjective algebra and always infinite-dimensional (except in the trivial case $A = 0$, which we exclude).

The underlying vectorspace of $\hat{A}$ is given by

$$\hat{A} = \left(\bigoplus_{i\in\mathbb{Z}} A\right) \oplus \left(\bigoplus_{i\in\mathbb{Z}} Q\right)$$

We denote the elements of $\hat{A}$ by $(a_i, \varphi_i)_i$, where $a_i \in A$, $\varphi_i \in Q$, of course with almost all a_i, φ_i being zero. The multiplication is defined by

$$(a_i, \varphi_i)_i \cdot (b_i, \psi_i)_i = (a_i b_i, a_{i+1}\psi_i + \varphi_i b_i)_i.$$

We refer to [H1] for an interpretation of $\hat{A}$ as doubly infinite matrix algebra.

We define an $\hat{A}$-module X as a sequence $X = (X_n, f_n)$ of A-modules X_n and A-linear maps $f_n : X_n \to \mathrm{Hom}_A(Q, X_{n+1})$ satisfying $f_{n-1}\cdot\mathrm{Hom}_A(Q, f_n) = 0$ for all $n \in \mathbb{Z}$. Instead of (X_n, f_n) we also write

$$\cdots X_{-2} \overset{f_{-2}}{\curvearrowright} X_{-1} \overset{f_{-1}}{\curvearrowright} X_0 \overset{f_0}{\curvearrowright} X_1 \overset{f_1}{\curvearrowright} X_2 \cdots.$$

A morphism $h : X = (X_n, f_n) \to Y = (Y_n, g_n)$ is a sequence $h = (h_n)$ of A-linear maps $h_n : X_n \to Y_n$ such that the following diagrams commute for all $n \in \mathbb{Z}$.

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & \mathrm{Hom}_A(Q, X_{n+1}) \\ {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_A(Q, h_{n+1})} \\ Y_n & \xrightarrow{g_n} & \mathrm{Hom}_A(Q, Y_{n+1}) \end{array}$$

We denote by $\mathrm{mod}\,\hat{A}$ the category of all $\hat{A}$-modules $X = (X_n, f_n)$ such that $\dim_k(\bigoplus_n X_n) < \infty$.

It is quite easy to see that $\mathrm{mod}\,\hat{A}$ is a Frobenius category in the sense of [H1]. In fact, the indecomposable projective-injective $\hat{A}$-modules are given by

$$\cdots 0 \curvearrowright X_i \overset{f_i}{\curvearrowright} X_{i+1} \curvearrowright 0 \cdots,$$

where X_{i+1} is an indecomposable A-injective module, $X_i = \mathrm{Hom}_A(Q, X_{i+1})$, and $f_i = id_{X_i}$. By $\underline{\mathrm{mod}}\,\hat{A}$ we denote the associated stable category. Thus the objects in $\underline{\mathrm{mod}}\,\hat{A}$ coincide with the objects in $\mathrm{mod}\,\hat{A}$, while the morphisms are given by $\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}(X, Y)/I(X, Y)$ where for $\hat{A}$-modules X, Y we have denoted by $I(X, Y)$ the subspace of those morphisms which factor over an injective $\hat{A}$-module.

It was shown in [H1] that $\underline{\mathrm{mod}}\,\hat{A}$ is a triangulated category, where the suspension functor serves as the translation functor. We will denote its application on a module $X \in \underline{\mathrm{mod}}\,\hat{A}$ by $X[1]$.

2.2. Let us now relate this description to the one given in the introduction.

The trivial extension algebra $T(A)$ of A by Q is the following finite-dimensional k-algebra. The additive structure is $A \oplus Q$ and the multiplication is defined by

$$(a, \varphi) \cdot (b, \psi) = (ab, a\psi + \varphi b)$$

for $a, b \in A$ and $\varphi, \psi \in Q$.

The algebra $T(A)$ is a $\mathbb{Z}$-graded algebra, where the elements of $A \oplus 0$ are the elements of degree 0 and those of $0 \oplus Q$ the elements of degree 1. We denote by $\mathrm{mod}^{\mathbb{Z}}\, T(A)$ the category of finitely generated $\mathbb{Z}$-graded $T(A)$-modules with morphisms of degree zero. The following is a straightforward observation.

The categories $\mathrm{mod}\,\hat{A}$ and $\mathrm{mod}^{\mathbb{Z}}\, T(A)$ are equivalent.

2.3. Now we are able to prove the theorem mentioned in the introduction.

Theorem. *Let A be a finite-dimensional k-algebra. $D^b(A)$ is triangle-equivalent to $\underline{\mathrm{mod}}\,\hat{A}$ if and only if gl.dim $A < \infty$.*

Proof. If gl.dim $A < \infty$ a proof was given in [H1].

For the converse suppose that $G : D^b(A) \to \underline{\mathrm{mod}}\,\hat{A}$ is a triangle-equivalence. By [AR] we infer that $\underline{\mathrm{mod}}\,\hat{A}$ has Auslander-Reiten triangles. Thus $D^b(A)$ has Auslander-Reiten triangles. So the assertion now follows from 1.5.

2.4. We have a canonical embedding ϕ of $\mathrm{mod}\,A$ into $\mathrm{mod}\,\hat{A}$ which sends a module $X \in \mathrm{mod}\,A$ onto (X_n, f_n) where $X_0 = X$ and $X_n = 0$ for $n \neq 0$. We infer that ϕ is exact.

The following two facts were established in [H2].

The composition of ϕ with the canonical functor $\mathrm{mod}\,\hat{A} \to \underline{\mathrm{mod}}\,\hat{A}$ is a full embedding.

Moreover we have for $X, Y \in \mathrm{mod}\,A$ and $i \in \mathbb{Z}$ that

$$\underline{\mathrm{Hom}}(\phi(X), (\phi(Y))[i]) \simeq \mathrm{Ext}^i_A(X, Y)$$

2.5. In [H1], (see also [H2]) we have constructed a full and faithful exact functor F of triangulated categories $F : D^b(A) \to \underline{\mathrm{mod}}\,\hat{A}$ such that F extends the identity functor on $\mathrm{mod}\,A$. A considerable simplification was obtained in [KV]. We will now give a very easy construction.

Theorem. *Let A be a finite-dimensional k-algebra. Then there exists a full and faithful exact functor F of triangulated categories $F : D^b(A) \to \operatorname{mod}\hat{A}$ such that F extends the identity functor on $\operatorname{mod}A$.*

Proof. Let $\phi : \operatorname{mod}A \to \operatorname{mod}\hat{A}$ be the exact functor defined in 2.4. Then ϕ extends to an exact functor $\tilde{\phi} : D^b(A) \to D^b(\hat{A})$. Now consider the localisation sequence as constructed in [Ri2]:

$$K^b({}_{\hat{A}}\mathscr{P}) \xrightarrow{\mu} D^b(\hat{A}) \xrightarrow{\pi} \underline{\operatorname{mod}}\,\hat{A}.$$

Set $F = \pi\tilde{\phi}$. Then F is an exact functor, which extends the identity functor on $\operatorname{mod}A$. It follows from 2.4. that F is full and faithful by using the Beilinson lemma [B].

Acknowledgment

These results were obtained while the author was visiting the University of Trondheim. And I would like to thank I. Reiten for the invitation and both she and M. Auslander for the discussions which provoked these investigations.

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