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COMAXIMIZABLE PRIMES

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ABSTRACT. Let $P_1, \dots, P_n$ $(n \geq 2)$ be not necessarily distinct nonzero prime ideals in the Noetherian, but not Henselian, domain R. We show that there is a finitely generated integral extension domain T of R, containing distinct, pairwise comaximal prime ideals $Q_1, \dots, Q_n$ lying over $P_1, \dots, P_n$ respectively.

1. INTRODUCTION

In [HW, §1], W. Heinzer and S. Wiegand give a pleasant proof of the following interesting result.

(1.1) **Theorem.** *Let P be a nonzero prime ideal in a Noetherian domain R, and let T be the integral closure of R in the algebraic closure of the quotient field of R. Let m be the number of primes in T that lie over P. Then either $m = \infty$ or $m = 1$. Furthermore, $m = 1$ if and only if R is a (local) Henselian domain, and P is its unique maximal ideal.*

The key step in their proof is the construction (in the case $m \neq 1$) of an integral extension domain of R in which two distinct primes, Q_1 and Q_2, lie over P. We show that when R is not Henselian, it is possible to force Q_1 and Q_2 to be comaximal. More generally, we consider the following idea.

Definition. Let $n \geq 2$, and let $P_1, \dots, P_n$ be not necessarily distinct nonzero prime ideals in the domain R. We say that $P_1, \dots, P_n$ are comaximizable if there is a finitely generated integral extension domain T of R, containing distinct, pairwise comaximal primes $Q_1, \dots, Q_n$ lying over $P_1, \dots, P_n$ respectively.

We prove the following theorem.

(1.2) **Theorem.** *For $n \geq 2$, let $P_1, \dots, P_n$ be not necessarily distinct nonzero prime ideals in the Noetherian domain R. Then $P_1, \dots, P_n$ are comaximizable if and only if R is not a (local) Henselian domain.*

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Throughout this paper, R will be a domain, and X will be an indeterminate over R. A domain with a unique maximal ideal will be called quasi-local. If it is also known to be Noetherian, we will call it local. We will freely use facts about integral extensions as in [K, §1-6]. For background on Henselian domains, see [N, §43]. We specifically note the following terminology. When we say a domain is not Henselian, we will mean that it is not a quasi-local Henselian domain. In particular, a domain which is not quasi-local is automatically not Henselian.

2. Preliminaries

Definition. Let P be a prime ideal in R, and let Q be a prime ideal in $R[X]$ with $Q \cap R = P$, but with $Q \neq PR[X]$. Then we will call Q an upper to P in $R[X]$ (or more simply, an upper to P, or just an upper). (See [K, §1-5].) If the upper Q contains a monic polynomial, we will call Q an integral upper.

Notice that if K is an upper to zero in $R[X]$, then there is a natural embedding $R \subseteq R[X]/K$, making the domain $R[X]/K$ a simple extension of R. If K is an integral upper, then $R[X]/K$ is a simple integral extension domain of R since K contains a monic polynomial.

(2.1) *Remark.* Let R be an integrally closed domain. Then the integral uppers to zero in $R[X]$ are exactly the primes K in $R[X]$ such that $K = f(X)R[X]$, with $f(X)$ a monic irreducible polynomial in $R[X]$. This fact is well known, and is easily proved starting with the (well-known) fact that if α is an element in some larger domain, and α is integral over R, then the minimal (monic irreducible) polynomial of α over the quotient field of R is actually in $R[X]$.

(2.2) **Lemma.** *Let $R \subseteq T$ be an integral extension of domains. If L is an upper to zero in $T[X]$, then $K = L \cap R[X]$ is an upper to zero in $R[X]$. If L is an integral upper, then K is an integral upper.*

Proof. As $R[X] \subseteq T[X]$ is an integral extension, and as $L \neq 0$, we have $K \neq 0$. Since $L \cap T = 0$, we have $K \cap R = 0$. Therefore, K is an upper to zero in $R[X]$. Now suppose that L is an integral upper. Then $T \subseteq T[X]/L$ is an integral extension. Thus $R \subseteq T[X]/L$ is an integral extension. Since $R \subseteq R[X]/K \subseteq T[X]/L$, we see that $R \subseteq R[X]/K$ is an integral extension, and thus K must contain a monic polynomial and be an integral upper.

(2.3) **Lemma.** *Let $g(X) \in B \in \operatorname{Spec} R[X]$, with R a domain and with $g(X)$ a monic polynomial in $R[X]$. Then there is an upper to zero, K, in $R[X]$, with $g(x) \in K \subseteq B$.*

Proof. Let $\overline{R}$ be the integral closure of R, and let $\overline{B}$ be a prime of $\overline{R}[X]$ lying over B. Let F be the quotient field of R, and let $g(X) = g_1(X) \cdots g_n(X)$ be the prime factorization of $g(X)$ in $F[X]$. If u_i is a root of $g_i(X)$ ($i = 1, \ldots, n$), then $g(u_i) = 0$, so that u_i is integral over $\overline{R}$. As $g_i(X)$ is the minimal polynomial of u_i over F, and as $\overline{R}$ is integrally closed, we know that $g_i(X) \in \overline{R}[X]$. Since $g(X) \in \overline{B}$, for some i we must have $g_i(X) \in \overline{B}$. By

(2.1), $g_i(X)\overline{R}[X]$ is an upper to zero in $\overline{R}[X]$ and is contained in $\overline{B}$. By (2.2), $g_i(X)\overline{R}[X] \cap R[X]$ is an upper to zero in $R[X]$, which clearly contains $g(X)$ and is contained in B. Let $K = g_i(X)\overline{R}[X] \cap R[X]$.

For the sake of reference, we state the next well-known lemma.

(2.4) **Lemma.** *Let S be a multiplicatively closed subset of the domain R, and let the domain D be an integral extension of R_S. Then there is an integral extension domain T of R, with $T_S = D$.*

(2.5) **Lemma.** *Let R be the integral closure of a Noetherian domain, and let L be a prime ideal of R. Then the integral closure of R/L is the integral closure of a Noetherian domain* (*and so is a Krull domain*).

Proof. Suppose that R is the integral closure of the Noetherian domain S. Let $R^\# = R/L$, and let $S^\# = S/(L \cap S)$. By [N, (33.10)], $R^\#$ is an almost finite integral extension of $S^\#$. (That is, the quotient field of $R^\#$ is finite-dimensional over the quotient field of $S^\#$.) Clearly we can find a domain $T^\#$ with $S^\# \subseteq T^\# \subseteq R^\#$, with $T^\#$ finitely generated over $S^\#$, and with the quotient field of $T^\#$ equal to the quotient field of $R^\#$. Thus $R^\#$ and $T^\#$ have the same integral closure. As $T^\#$ is a Noetherian domain, we are done.

(2.6) **Lemma.** *Let R be a domain that is not a field, and suppose that $\overline{R}$, the integral closure of R, is a Krull domain. Then in $R[X]$, there is a monic irreducible polynomial $\alpha(X)$, with degree $\alpha(X) \geq 2$.*

Proof. Since $\overline{R}$ is not a field, it contains some $b \neq 0$ with b a nonunit. Since $\overline{R}$ is a Krull domain, it enjoys the ascending chain condition on principal ideals. Therefore, we may choose our b such that $b\overline{R}$ is as large as possible. Clearly b is irreducible in $\overline{R}$, and hence is not the square of any element in $\overline{R}$. Thus $X^2 - b$ is irreducible in $\overline{R}[X]$. Therefore, by (2.1), $L = (X^2 - b)\overline{R}[X]$ is an integral upper to zero in $\overline{R}[X]$. By (2.2), $L \cap R[X]$ is an integral upper to zero in $R[X]$. Let $\alpha(X)$ be a monic polynomial in $L \cap R[X]$, with degree $\alpha(X)$ as small as possible. Then clearly $\alpha(X)$ is irreducible in $R[X]$. Also, we must have degree $\alpha(X) \geq 2$, since $X^2 - b$ divides $\alpha(X)$ in $\overline{R}[X]$. Thus $\alpha(X)$ satisfies the lemma.

(2.7) **Lemma.** *Let R be the integral closure of a Noetherian domain, and let P be a nonzero prime ideal of R. Then the intersection of all powers of P is zero.*

Proof. Let R be the integral closure of the Noetherian domain S, and let u be an element in P but in none of the other (finitely many) primes of R that lie over $P \cap S$ [N, (33.10)]. Then we easily see that P is the only prime of R lying over $P \cap S[u]$. A well-known theorem of Chevalley [C] says that there is a discrete valuation ring (V, N) between $S[u]$ and its quotient field, with $N \cap S[u] = P \cap S[u]$. Clearly $R \subseteq V$, and $N \cap R$ must equal P. As the intersection of all powers of N is zero, the same must be true of the intersection of all powers of P.

The next lemma is a key result of [HW, §1]. We offer a variation on the proof.

(2.8) **Lemma** [HW, (1.4)]. *Let R be an integrally closed domain, and suppose that N and H are distinct maximal ideals of R, with $H \neq H^2$. Then there is a simple integral extension domain of R in which two distinct maximal ideals lie over N.*

Proof. As N and H are comaximal, pick $a \in R$ with $a - 1 \in N$ and $a \in H$. Let $y \in H - H^2$. Since N and H^2 are comaximal, pick $b \in R$ with $b \in N$ and $b - y \in H^2$. Note that $b \in H - H^2$. Let $f(X) = X^2 - aX + b$. We claim that $f(X)$ is irreducible in $R[X]$. If not, it would have a root $\alpha \in R$. Since $\alpha^2 = a\alpha - b$ and $a, b \in H$, we have $\alpha \in H$. Since $b = a\alpha - \alpha^2$, and $a, \alpha \in H$, we have $b \in H^2$. This is a contradiction, which proves that $f(X)$ is irreducible. Let $K = f(X)R[X]$, so that K is an integral upper to zero, by (2.1). Thus $R[X]/K$ is a simple integral extension domain of R. We claim that it contains two maximal ideals lying over N. Since $a \equiv 1 \mod N$, and $b \in N$, $f(X) \equiv X^2 - X \mod N$. Thus $K = f(X)R[X]$ is contained in $(N, X)R[X]$ and in $(N, X-1)R[X]$. One can easily verify that $(N, X)R[X]/K$, and $(N, X-1)R[X]/K$ are distinct maximal ideals in $R[X]/K$, both of which lie over N.

(2.9) **Corollary.** *Let R be Noetherian domain that is not Henselian. Let N be a maximal ideal of $\overline{R}$, the integral closure of R. Then $\overline{R}_N$ is not Henselian.*

Proof. First, suppose that N is the unique maximal ideal of $\overline{R}$, so that $\overline{R} = \overline{R}_N$. Since the local domain R is not Henselian, there is an integral extension domain T of R that is not quasi-local. Consider the composite $T\overline{R}$. Being an integral extension of T, it cannot be quasi-local. As it is also an integral extension of $\overline{R}$, we have that $\overline{R} = \overline{R}_N$ is not Henselian in this case. Next, suppose that H is a maximal ideal of $\overline{R}$ distinct from N. By (2.7), $H^2 \neq H$. By (2.8), there is an integral extension domain of $\overline{R}$ in which two distinct maximal ideals lie over N. Localizing, we see that there is an integral extension domain of $\overline{R}_N$ in which two distinct maximal ideals lie over N_N. Again, $\overline{R}_N$ is not Henselian.

3. Comaximizable primes

(3.1) **Proposition.** *Let R be a Noetherian domain. Let P and Q be nonzero primes of R, and suppose there is a maximal ideal M of R with $P \subseteq M$ but $Q \nsubseteq M$. Then there is a simple integral extension domain R' of R, and primes P' and Q' of R' lying over P and Q respectively, such that P' and Q' are comaximal in R'.*

Proof. Suppose that in $R[X]$, we can find an integral upper K to zero, with $K \subseteq (P, X) \cap (Q, X-1)$. Then $R' = R[X]/K$ will be a simple integral extension of R, and $P' = (P, X)/K$ and $Q' = (Q, X-1)/K$ will be primes of R' lying over P and Q. Since P' will contain the coset X mod K, and Q' will contain the coset $X-1$ mod K, no ideal of R' can contain both P' and Q', since such an ideal would then contain the coset 1 mod K, which is the identity of R'. Therefore, we need only find such a K.

We now lift our problem to the integral closure $\overline{R}$ of R. Let $\overline{P}$ and $\overline{Q}$ be primes of $\overline{R}$ lying over P and Q respectively. By the going up theorem, there is a maximal ideal $\overline{M}$ of $\overline{R}$ lying over M with $\overline{P} \subseteq \overline{M}$. Clearly $\overline{Q} \nsubseteq \overline{M}$. Thus the hypotheses on R, P, Q, and M lift to $\overline{R}$, $\overline{P}$, $\overline{Q}$, and $\overline{M}$, except that the hypothesis that R is Noetherian is replaced with the fact that $\overline{R}$ is the integral closure of a Noetherian domain. Suppose now that we can find an integral upper $\overline{K}$ to zero in $\overline{R}[X]$, with $\overline{K} \subseteq (\overline{P}, X) \cap (\overline{Q}, X-1)$. Let $K = \overline{K} \cap R[X]$. By (2.2), K will be an integral upper to zero in $R[X]$, and obviously K will be contained in $(P, X) \cap (Q, X-1)$. Thus, we need only find $\overline{K}$.

We will return to our original unbarred notation, now assuming that R is the integral closure of a Noetherian domain. By (2.1), we seek a monic polynomial $f(X)$ that is irreducible in $R[X]$, and satisfies $f(X) \in (P, X) \cap (Q, X-1)$. Since Q and M are comaximal, we can find an $a \in R$ with $a \equiv 0 \bmod M$ and $a \equiv 1 \bmod Q$. Thus $(Q, X-1) = (Q, X-a)$, and $a \in M$. Since (2.7) shows that the intersection of all powers of M is zero, clearly $MP \neq P$; thus $P = MP + QP$ implies $QP \nsubseteq MP$. Choose $b \in QP - MP$. Let $f(X) = X^2 - aX + b$. As $b \in P$, clearly $f(X) \in (P, X)$. As $b \in Q$, clearly $f(X) \in (Q, X-a) = (Q, X-1)$. It only remains to show that $f(X)$ is irreducible in $R[X]$. If not, then write $f(X) = (X-c)(X-d)$. Since $cd = b \in P$, we may assume that $c \in P$. Now $a, c \in M$, and $a = c + d$ implies $d \in M$. Thus $b = cd \in MP$, which contradicts our choice of b. This shows that $f(X)$ is irreducible, and completes the proof.

(3.2) *Remark.* In (3.1), we can also arrange to have height $P' = \operatorname{height} P$, and $\operatorname{height} Q' = \operatorname{height} Q$. We will show how to get the latter, the former working similarly. Since $Q' = (Q, X-1)/K$, we want height $(Q, X-1)/K = \operatorname{height} Q$. Recall that via the second paragraph of the above proof, $K = \overline{K} \cap R[X]$. Thus $R \subseteq R[X]/K \subseteq \overline{R}[X]/\overline{K}$ are integral extensions. Since $(\overline{Q}, X-1)/\overline{K}$ lies over $(Q, X-1)/K$, which in turn lies over Q, we always have $\operatorname{height} Q \geq \operatorname{height}(Q, X-1)/K \geq \operatorname{height}(\overline{Q}, X-1)/\overline{K}$. If we can arrange to have $\operatorname{height}(\overline{Q}, X-1)/\overline{K} = \operatorname{height} Q$, equality will hold throughout, and we will be done. However, since $\overline{R}$ is integrally closed, the integral extension $\overline{R} \subseteq \overline{R}[X]/\overline{K}$ satisfies the going down theorem, and so, since $(\overline{Q}, X-1)/\overline{K}$ lies over $\overline{Q}$, these two primes have the same height. Thus, we only need $\operatorname{height} \overline{Q} = \operatorname{height} Q$. Of course we know this does hold for some $\overline{Q}$ lying over Q, completing the argument.

(3.3) **Corollary.** *Let R be a Noetherian domain that is not Henselian. Let M be a maximal ideal of R with* $\operatorname{height} M \geq 2$. *Let $P_1, \ldots, P_n$ be nonzero prime ideals in R. Then there is a finitely generated integral extension domain R'*

of R, and primes M' and $P_1', \ldots, P_n'$ in R' lying over M and $P_1, \ldots, P_n$ respectively, such that height $M' \geq 2$, *and $P_i' \not\subseteq M'$, for $i = 1, \ldots, n$.*

Proof. We will induct on n. Suppose $n = 1$, and let $P = P_1$. If $P \not\subseteq M$, we let $R' = R$, and we are done. Suppose that $P \subseteq M$. Let $\overline{R}$ be the integral closure of R, and let N be a maximal ideal of $\overline{R}$ lying over M, with height $N =$ height $M \geq 2$. By (2.9), since R is not Henselian, neither is $\overline{R}_N$. Thus there is an integral extension domain of $\overline{R}_N$ having at least two distinct maximal ideals lying over N_N, and so by (2.4) there is an integral extension domain T' of $\overline{R}$ having at least two distinct maximal ideals, M_1' and M_2', lying over N. Since $\overline{R}$ is integrally closed, the going down theorem shows that for $i = 1, 2$, height $M_i' =$ height $N \geq 2$. Now pick $u \in M_1' - M_2'$. Let $T = R[u]$, $M_1 = M_1' \cap T$, and $M_2 = M_2' \cap T$. Then T is Noetherian, M_1 and M_2 lie over M in R, M_1 and M_2 are distinct (since $u \in M_1 - M_2$), and M_1 and M_2 both have height at least 2. Let p be a prime of T lying over P. If $p \not\subseteq M_1$, then we let $R' = T$, $P' = p$, and $M' = M_1$, and we are done. Therefore, suppose that $p \subseteq M_1$. Then by (3.1) (applied in T, with $P = p$, $Q = M_2$, and $M = M_1$), there is a finitely generated integral extension domain R' of T, and comaximal primes P' and M' of R' lying over p and M_2 respectively. By (3.2), we may assume that height $M' =$ height $M_2 \geq 2$. Since P' and M' are comaximal, surely $P' \not\subseteq M'$. This R', P', and M' satisfy the corollary (when $n = 1$).

Suppose now that for $n > 1$, we know the corollary is true for $n - 1$. Then applying the case $n - 1$ to M and $P_1, \ldots, P_{n-1}$, we can find a finitely generated integral extension domain $R^{\#}$ of R, and primes $M^{\#}$, $P_1^{\#}, \ldots, P_{n-1}^{\#}$ lying over M, $P_1, \ldots, P_{n-1}$ respectively, with height $M^{\#} \geq 2$ and $P_i^{\#} \not\subseteq M^{\#}$ for $i = 1, \ldots, n-1$. Let $P_n^{\#}$ be any prime of $R^{\#}$ lying over P_n. Apply the case $n = 1$ to $R^{\#}$, $M^{\#}$, and $P_n^{\#}$, and find a finitely generated integral extension domain R' of $R^{\#}$, with primes M' and P_n' lying over $M^{\#}$ and $P_n^{\#}$, and with height $M' \geq 2$ and $P_n' \not\subseteq M'$. Now let $P_1', \ldots, P_{n-1}'$ be any primes of R' lying over $P_1^{\#}, \ldots, P_{n-1}^{\#}$ respectively. Since for $i = 1, \ldots, n-1$ we have $P_i^{\#} \not\subseteq M^{\#}$, surely we must have $P_i' \not\subseteq M'$. The corollary is clearly satisfied, and we are done.

(3.4) **Proposition.** *Let R be a Noetherian domain. Let P and Q be nonzero prime ideals of R. (Here, we allow the possibility that $P = Q$.) Suppose there is a maximal ideal M of R such that $P \not\subseteq M$, $Q \not\subseteq M$, and* height $M \geq 2$. *Then there is a simple integral extension domain R' of R, and distinct primes P' and Q' of R' lying over P and Q respectively, such that P' and Q' are comaximal in R'.*

Proof. As in the proof of (3.1), we need only find an integral upper K to zero in $R[X]$ with $K \subseteq (P, X) \cap (Q, X - 1)$. As in that proof, we may also lift to the integral closure of R (here using the fact that the integral closure of R

contains a maximal ideal lying over M and having the same height as M). Thus, we will assume that R is the integral closure of a Noetherian domain. By (2.1), we seek a monic polynomial $f(X)$ that is irreducible in $R[X]$, and satisfies $f(X) \in (P, X) \cap (Q, X-1)$.

Since $P \not\subseteq M$ and $Q \not\subseteq M$, we have M and $P \cap Q$ comaximal. Write $1 = x + y$, with $x \in P \cap Q$ and $y \in M$. Let L be a prime of R minimal over y, with $L \subseteq M$. Since $1 = x + y$, we see that L and $P \cap Q$ are comaximal. Since R is a Krull domain, $\operatorname{height} L = 1 < \operatorname{height} M$, so that L is not maximal. Thus R/L is not a field, and by (2.5), its integral closure is a Krull domain. Using (2.6), let $\alpha(x) = X^m + \alpha_{m-1}X^{m-1} + \cdots + \alpha_0$ be a monic irreducible polynomial in $(R/L)[X]$ with $m \geq 2$. For $i = 0, 1, \ldots, m-1$, let $b_i \in R$ be a coset representative of α_i. Since L and $P \cap Q$ are comaximal, for $i = 0, 1, \ldots, m-1$, use the Chinese Remainder Theorem to find $c_i \in R$ with $c_i \equiv b_i \bmod L$, and $c_i \equiv 0 \bmod P \cap Q$. Thus $c_i \in P \cap Q$, and c_i is a coset representative for α_i. Since L and Q are clearly comaximal, pick $a \in R$, with $a \equiv 0 \bmod L$ and $a \equiv 1 \bmod Q$. Thus $a \in L$ and $(Q, X-1) = (Q, X-a)$. We now define $f(X)$ to be $X^{m-1}(X-a) + c_{m-1}X^{m-1} + \cdots + c_0$. Since each $c_i \in P$, and since $m - 1 \geq 1$, we see immediately that $f(X) \in (P, X)$. Since each $c_i \in Q$, we also see that $f(X) \in (Q, X-a) = (Q, X-1)$. It only remains to show that $f(X)$ is irreducible in $R[X]$. Surely it will suffice to show that $f(X)$ taken modulo L is irreducible in $(R/L)[X]$. However, we claim that $f(X) \bmod L$ is just the irreducible polynomial $\alpha(X)$. This is easily seen, since $a \in L$, and since for $i = 0, \ldots, m-1$, $c_i \bmod L$ is α_i. This completes the proof.

We now prove our main result, (1.2).

Proof of (1.2). Since any integral extension domain of a Henselian domain is quasi-local, one direction is obvious. For the other direction, assume that R is not Henselian. We first treat the case in which R contains a maximal ideal M, with height $M \geq 2$. By (3.3), it does no harm to assume that M does not contain any of $P_1, \ldots, P_n$. Fix an algebraic closure Ω of the quotient field of R and without loss, assume that all extensions of R that we mention come from Ω. Consider all possible ordered $n+1$-tuples $(T, Q_1, \ldots, Q_n)$ where T is a finitely generated integral extension domain of R, and $Q_1, \ldots, Q_n$ are primes of T lying over $P_1, \ldots, P_n$ respectively. Our task is to find such an $n+1$-tuple such that whenever $i \neq k$, we have Q_i and Q_k comaximal (and hence distinct) in T. For each such $n+1$-tuple, consider the set $\{(i, k) \mid Q_i$ and Q_k are comaximal in $T\}$. Let $(T, Q_1, \ldots, Q_n)$ be a fixed $n+1$-tuple chosen so as to make the size of the set just mentioned as large as possible. We claim that in fact Q_i and Q_k are comaximal whenever $i \neq k$, so that this $n+1$-tuple is the one we seek. Assume this is not so. We will derive a contradiction. For some $i \neq k$, Q_i and Q_k are assumed not comaximal. Reordering if necessary, we may say that Q_1 and Q_2 are not comaximal. Let N be a maximal ideal of T lying over M, with $\operatorname{height} N = \operatorname{height} M \geq 2$.

Since M does not contain P_1 or P_2, clearly N does not contain Q_1 or Q_2. By (3.4), there is a finitely generated integral extension domain T' of T, with primes Q_1' and Q_2' lying over Q_1 and Q_2 respectively, and with Q_1' and Q_2' comaximal. Let $Q_3', \ldots, Q_n'$ be any primes of T' lying over $Q_3, \ldots, Q_n$ respectively. Note that if for some $i \neq k$ we have Q_i and Q_k comaximal in T, then automatically we have Q_i' and Q_k' comaximal in T'. Thus, $\{(i, k) \mid Q_i$ and Q_k are comaximal in $T\} \subseteq \{(i, k) \mid Q_i'$ and Q_k' are comaximal in $T'\}$. By the maximality of our choice $(T, Q_1, \ldots, Q_n)$, we must have that these two sets are equal. On the other hand, we have $(1, 2)$ in the second of these sets, but not in the first. This contradiction completes the argument for this case.

The remaining case is when every maximal ideal of R has height 1. In particular, this means that each of the primes $P_1, \ldots, P_n$ is maximal. Let T^* be the integral closure of R in Ω. By (1.1), we see that in T^*, there are distinct primes $Q_1^*, \ldots, Q_n^*$ lying over $P_1, \ldots, P_n$ respectively. For $1 \leq i \leq n$, pick u_i in Q_i^* but in none of the rest of $Q_1^*, \ldots, Q_n^*$. Let $T = R[u_1, \ldots, u_n]$, and let $Q_i = Q_i^* \cap T$. We see that T is finitely generated over R, and that $Q_1, \ldots, Q_n$ are distinct primes of T lying over $P_1, \ldots, P_n$ respectively. Being distinct and maximal assures that $Q_1, \ldots, Q_n$ will be pairwise comaximal. Thus $P_1, \ldots, P_n$ are comaximizable, and we are done.

As a corollary, we strengthen (1.1) in the case that R is not Henselian.

(3.5) **Corollary.** *Let P be a nonzero prime ideal in the Noetherian, but not Henselian, domain R. Let T be the integral closure of R in the algebraic closure of the quotient field of R. Then in T, there is an infinite set of distinct, pairwise comaximal primes $P_1, P_2, P_3, \ldots$, all lying over P.*

Proof. We will inductively construct a chain of rings, $R \subseteq R_1 \subseteq R_2 \subseteq R_3 \subseteq \cdots \subseteq T$, such that for $m \geq 1$, R_m will be a finitely generated extension of R, and will contain distinct primes $Q_{m1}, \ldots, Q_{mm}$ all lying over P. Furthermore, if $m \geq 2$, then $Q_{m1}, \ldots, Q_{mm}$ will be pairwise comaximal, and we will have $Q_{mk} \cap R_{m-1} = Q_{(m-1)k}$ for $1 \leq k \leq m-1$, while $Q_{mm} \cap R_{m-1} = Q_{(m-1)(m-1)}$.

Suppose that this chain has been constructed. Let $D = \bigcup R_m$, $1 \leq m \leq \infty$. Also, for any $m \geq 1$, let $Q_m = \bigcup Q_{im}$, $m \leq i \leq \infty$. The reader can easily verify that D is a domain between R and T, and that $Q_1, Q_2, Q_3, \ldots$ are distinct, pairwise comaximal, and lie over P. Thus, if $P_1, P_2, P_3, \ldots$ are any primes of T lying over $Q_1, Q_2, Q_3, \ldots$ respectively, then $P_1, P_2, P_3, \ldots$ are seen to satisfy the conclusion of the corollary.

It only remains to construct the R_m and $Q_{m1}, \ldots, Q_{mm}$. Let $R_1 = R$ and $Q_{11} = P$, and inductively suppose that for $m-1 \geq 1$, we already have R_{m-1}, and $Q_{(m-1)1}, \ldots, Q_{(m-1)(m-1)}$. Since R_{m-1} is finitely generated over R, it is Noetherian. We claim that R_{m-1} is not Henselian. If $m-1 = 1$, it is given that $R_1 = R$ is not Henselian. If $m-1 \geq 2$, then since $Q_{(m-1)1}, \ldots, Q_{(m-1)(m-1)}$ are distinct and pairwise comaximal, clearly R_{m-1}

is not local, and so is not Henselian. We apply (1.2) to R_{m-1} and the list of primes $Q_{(m-1)1}, Q_{(m-1)2}, \ldots, Q_{(m-1)(m-2)}, Q_{(m-1)(m-1)}, Q_{(m-1)(m-1)}$. (Here, note that the last two primes in our list are identical.) By (1.2), there is a finitely generated integral extension domain R_m of R_{m-1} and a list of distinct, pairwise comaximal primes $Q_{m1}, \ldots, Q_{mm}$ of R_m lying respectively over the corresponding primes in our first list. This completes the proof.

We leave to the reader the proof of the next corollary, a variation on the Chinese Remainder Theorem.

(3.6) **Corollary.** *Let R be a Noetherian, but not Henselian, domain. Let $P_1, \ldots, P_n$ be not necessarily distinct nonzero prime ideals in R, and let $a_1, \ldots, a_n$ be elements of R. Then there is an irreducible monic polynomial $f(X)$ in $R[X]$, such that $f(a_i) \in P_i$ for $1 \leq i \leq n$.*

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