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A HYPERCYCLIC OPERATOR WHOSE ADJOINT IS ALSO HYPERCYCLIC

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ABSTRACT. An operator T acting on a Hilbert space H is hypercyclic if, for some vector x in H, the orbit $\{T^n x: n \geq 0\}$ is dense in H. We show the existence of a hypercyclic operator—in fact, a bilateral weighted shift—whose adjoint is also hypercyclic. This answers positively a question of D. A. Herrero.

INTRODUCTION

Let H be a complex, separable, infinite-dimensional Hilbert space. Let $B(H)$ denote the bounded linear operators acting on H. Let $T \in B(H)$ and $x \in H$. The orbit of x under T is

$$\operatorname{Orb}(T, x) = \{x, Tx, T^2x, \dots\}.$$

T is said to be cyclic if there exists $y \in H$ such that the linear span of $\operatorname{Orb}(T, y)$ is dense in H; T is hypercyclic if $\operatorname{Orb}(T, y)$ itself is dense in H. In this last case, y is hypercyclic for T (in [2] such a y is called universal, and in [4] such a y is called orbital).

In a recent paper [7], D. A. Herrero completely characterized the closure of the class of hypercyclic operators in $B(H)$ in terms of spectral properties. His proof is based on previous work of C. Kitai [10], Godefroy and Shapiro [3], and Herrero [5] and [6]. Also a key result that he uses is the similarity orbit theorem [1]. In [7], Herrero raises several questions. The purpose of this note is to answer positively Question 2: "Does there exist $T \in B(H)$ such that both T and T^* are hypercyclic?" We also answer in the affirmative Question 3: "Does there exist a hypercyclic backward shift such that $b_0 b_1 \cdots b_{k-1}$ does not tend to infinity? (Here the b_i's are the weights of the shift.)"

Gethner and Shapiro proved in [2] that if the weights $b_k \geq 1$ and $b_0 \cdots b_k$ go to infinity, then the backward shift is hypercyclic. In [7], it is shown that the condition $b_k \geq 1$ can be dropped. Observe that if U is the unilateral (unweighted) shift, $2U^*$ is hypercyclic, while $2U$ is not.

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THE EXAMPLE

Let $\{e_k : k \in Z\}$ be an orthonormal basis for the Hilbert space H. Let $S \in B(H)$ be defined by $Se_k = e_{k+1}$; i.e., S is a bilateral (unweighted) shift.

Assume that $\{z_k = \sum_{i=-p_k}^{p_k} z_k(i)e_i : k \in N\}^- = H$. For $\{n_i : i = \pm 1, \pm 2, \ldots\}$, to be defined later, and so that $\cdots < n_{-2} < n_{-1} < 0 < n_1 < n_2 < \ldots$, we define a sequence of weights:

$$(1) \qquad b_k = \begin{cases} 2 & \text{if } k \neq n_i, \\ 2^{-2n_{i+1}} & \text{if } k = n_{-i} \text{ and } i > 0, \\ 2^{2n_{-i}} & \text{if } k = n_i \text{ and } i > 0. \end{cases}$$

We will construct $T \in B(H)$ such that T and T^* are both hypercyclic; the polar decomposition of T will be

$$(2) \qquad T = S^* P, \quad \text{where } Pe_k = b_k e_k .$$

Let $-n_1 = n_{-1}$ be such that $n_1 > 2(p_1 + p_2)$ and $2^{-n_1+p_1+1}\|z_1\| < 1/2$. Let $x_1 = 2^{-n_1+p_1+1}S^{n_1-p_1-1}(z_1)$ and $y_1 = 2^{-n_1+p_1+1}(S^*)^{n_1-p_1-1}(z_1)$.

Assume that we have already chosen n_i, n_{-i}, x_i, and y_i for $1 \leq i \leq j$,

$$(3) \qquad x_i \in \operatorname{span}\{e_k : n_i - 2p_i - 1 \leq k \leq n_i - 1\},$$

$$(4) \qquad y_i \in \operatorname{span}\{e_k : n_{-i} + 1 \leq k \leq n_{-i} + 2p_i + 1\},$$

so that, for $1 < k \leq j$,

$$(5) \qquad n_k > 2\left(n_{k-1} + |n_{-k+1}| + \sum_{i=1}^{k+1} p_i\right),$$

$$(6) \qquad |n_{-k}| > 2\left(|n_{-k+1}| + n_k + \sum_{i=1}^{k+1} p_i\right),$$

$$(7) \qquad \|x_k\| < 2^{-2n_{k-1}},$$

$$(8) \qquad \|y_k\| < 2^{2n_{-k+1}}.$$

We are now ready to choose n_{j+1} and x_{j+1}; next we will choose n_{-j-1} and y_{j+1}. Recall that $b_k = 2$ if $n_j < k \leq n_j + 2p_{j+1}$. Set

$$(9) \qquad u_{j+1} = \sum_{i=-p_{j+1}}^{p_{j+1}} \left(\prod_{r=i+1}^{n_j+p_{j+1}+i} b_r \right)^{-1} z_{j+1}(i) e_{n_j+p_{j+1}+i}.$$

Let n_{j+1} be a positive integer satisfying

$$(10) \qquad n_{j+1} > 2\left(n_j + |n_{-j}| + \sum_{i=1}^{j+2} p_i\right),$$

$$(11) \qquad 2^{-n_{j+1}+2p_{j+1}}\|u_{j+1}\| < 2^{-3n_j-1}.$$

Set

$$x_{j+1} = 2^{-n_{j+1}+n_j+2p_{j+1}+1} S^{n_{j+1}-n_j-2p_{j+1}-1}(u_{j+1}). \tag{12}$$

Having defined n_{j+1} and x_{j+1}, we now proceed to define n_{-j-1} and y_{-j-1}. Let

$$v_{j+1} = \sum_{i=-p_{j+1}}^{p_{j+1}} \left(\prod_{r=n_{-j}-p_{j+1}+i}^{i} b_r \right)^{-1} z_{j+1}(i) e_{n_{-j}-p_{j+1}+i-1}. \tag{13}$$

Let n_{-j-1} be a negative integer satisfying

$$|n_{-j-1}| > 2\left(n_{j+1} + |n_{-j}| + \sum_{i=1}^{j+2} p_i \right) \tag{14}$$

$$2^{n_{-j-1}+2p_{j+1}} \|v_{j+1}\| < 2^{3n_{-j}-2}. \tag{15}$$

Set

$$y_{j+1} = 2^{n_{-j-1}-n_{-j}+2p_{j+1}+2} (S^*)^{-n_{-j-1}+n_{-j}-2p_{j+1}-2}(v_{j+1}). \tag{16}$$

Define $x = \sum_{k=1}^{\infty} x_k$ and $y = \sum_{k=1}^{\infty} y_k$. Since $\{n_i : i \in Z \backslash \{0\}\}$ has been specified, the weights are defined by (1), and T is defined by (2). We will show that x is hypercyclic for T and that y is hypercyclic for T^*. Observe that definitions (9) and (12) imply that

$$T^{n_j-p_j-1}(x_j) = z_j, \tag{17}$$

while definitions (13) and (16) imply that

$$(T^*)^{-n_{-j}-p_j-1}(y_j) = z_j. \tag{18}$$

Let $1 \leq k < j$. Definition (1) and inequality (5) imply that

$$\|T^{n_j-p_j-1}(x_k)\| < 2^{n_j-p_j} b_{n_{-j+1}} \|x_k\| < 2^{-n_j} \|x_k\|.$$

On the other hand, if $k > j$, inequality (11) and the definition of x_k (see (12)) imply that

$$\|T^{n_j-p_j-1}(x_k)\| < 2^{n_j-p_j} \|x_k\| < 2^{-2n_{k-1}+n_j}.$$

These last two estimates, together with (17), imply that

$$\begin{aligned} \|T^{n_j-p_j-1}(x) - z_j\| &< \sum_{k=1}^{j-1} \|T^{n_j-p_j-1}(x_k)\| + \sum_{k>j} \|T^{n_j-p_j-1}(x_k)\| \\ &< 2^{-n_j} \left(\sum_{k=1}^{j-1} \|x_k\| \right) + \sum_{k>j} 2^{-2n_{k-1}+n_j}. \end{aligned}$$

Thus, from inequalities (5) and (7), it follows that the last quantity converges to zero when j goes to infinity.

A similar argument (but using (6), (15), (16), and (18) instead of (5), (11), (12), and (17)) yields the result that $(T^*)^{-n_{-j}-p_j-1}(y) - z_j$ converges to zero when j goes to infinity. This concludes the proof that T and T^* are hypercyclic operators.

Remark 1. The compression of T to the subspace spanned by $\{e_k: k \geq 0\}$ answers Question 3 of [7] positively.

Remark 2. D. A. Herrero (personal communication) has observed the following:

(a) Since the representing matrix of T with respect to $\{e_i: i \in Z\}$ has real entries, an unpublished result of Deddens (see [8]) says that $T \oplus T^*$ is not hypercyclic; this answers Question 4 of [7] negatively.

(b) Up to now, all the hypercyclic operators have satisfied the Kitai-Gethner-Shapiro (K-G-S) condition: If $L \in B(H)$ has a right inverse M and a dense subset D of H so that $\|L^n(x)\|$ and $\|M^n(x)\|$ go to zero for every $x \in D$, then L has a hypercyclic vector (here H is separable, see [2, Theorem 2.2]). It is clear that if two operators satisfy the K-G-S condition, then the direct sum also satisfies it, and therefore the direct sum has to be hypercyclic. Thus either T or T^* does not satisfy the K-G-S condition.

Remark 3. The question "Is there any hypercyclic operator L such that $\operatorname{Per} L = \{x \in H: \operatorname{Orb}(L, x)$ is periodic$\}$ is a nontrivial finite dimensional space?" [7, Question 7] can also be answered positively. This question relates to [7, Proposition 4.7]. There it is shown that, if L is hypercyclic, either $(\operatorname{Per} L)^- = H$ or $(\operatorname{Per} L)^-$ has infinite codimension in H; it is also shown that $(\operatorname{Per} L)^- = H$ for all L in a dense subset of the class of hypercyclic operators.

There exists $A = \left(\begin{smallmatrix} 1 & F \\ 0 & G \end{smallmatrix}\right) \in B(C \oplus H)$, A hypercyclic, such that $\operatorname{Per} A = C \oplus \{0\}$. G is a unilateral weighted shift and F is a linear functional. (It is even possible to pick G so that G and G^* are both hypercyclic; however, A^* cannot be hypercyclic since A has an eigenvector (see [7]).) Herrero and Wang [9] have also answered Question 7 positively as a corollary of their proof of [7, Conjecture 1]. Originally, we did not plan to give details of the construction of A, but the referee suggested that it would be more useful to do so.

Let $\{e_n: n \geq 0\}$ be an orthogonal basis for H. Given $\{\beta_j \oplus z_j: z_j = \sum_{i=0}^{p_j} z_j(i)e_i,\ j \in N\}^- = C \oplus H$, it is possible to define inductively four sequences r_j, n_j, α_j, and g_j so that the functional F is represented by the vector

$$u = \sum_{j=1}^{\infty} \alpha_j e_{n_{j-1}+p_j+3},$$

while $G(e_0) = 0$ and $G(e_{i+1}) = g_i e_i$. The sequences r_j, n_j satisfy the relations

$$n_{j+1} = 2(r_{j+1} + p_{j+1}) + n_j + 3,$$

$$r_j > 3\left(\sum_{i \leq j-1} n_i + \sum_{i=1}^{j+1} p_i\right).$$

The weight sequence g_i satisfies:

$$g_i = \begin{cases} 2^{-2n_k} & \text{if } i = n_k \quad (n_0 = 0), \\ 1 & \text{if } n_k + 1 \le i \le 1 + n_k + p_{k+1}, \\ 1 & \text{if } 1 \le k \quad \text{and} \quad n_k - p_k < i \le n_k - 1, \\ 2^{-r_{k+1}} & \text{if } i = n_k + p_{k+1} + 2, \\ 2 & \text{otherwise}. \end{cases}$$

One can then choose a hypercyclic vector for A of the form

$$0 \oplus x = 0 \oplus \sum_{j=1}^{\infty} x_j,$$

where $x_j \in \operatorname{span}\{e_i : n_j - p_j \le i \le n_j\}$. The x_j's are constructed by using auxiliary vectors v_j. We show how x_1 and x_{j+1} are obtained from v_1 and v_{j+1}, respectively. This will put in evidence how r_j, n_j, and α_j are interrelated.

If $v = \sum_{i=0}^{m} v(i)e_i$, we denote $\sum_{i=0}^{m} v(i)$ by $h(v)$. The vector $v_1 = \sum_{i=0}^{p_1} v_1(i)e_i$ satisfies

$$\|v_1 - z_1\| \le 1/2,$$

and

$$h(v_1) \ne 0;$$

r_1 is chosen such that

$$2^{-r_1}\|v_1\| \le 1 \quad \text{and} \quad |\beta_1/(h(v_1)2^{r_1})| \le 1/2.$$

Then α_1 and x_1 satisfy

$$\alpha_1 = \beta_1/(h(v_1)2^{r_1}) \quad \text{and} \quad x_1 = 2^{-r_1} U^{n_1 - p_1}(v_1),$$

where U is the unilateral shift with respect to $\{e_i : i \ge 0\}$. The vector $v_{k+1} \in \operatorname{span}\{e_i : n_k + 2 \le i \le n_k + p_{k+1} + 2\}$ and is such that

$$\|G^{n_k+2}(v_{k+1}) - z_{k+1}\| \le 2^{-k-1} \quad \text{and} \quad h(v_{k+1}) \ne 0.$$

The numbers r_{k+1} satisfy

$$\|v_{k+1}\| \le 2^{-2n_k + r_{k+1}},$$

and if

$$\alpha_{k+1} = (\beta_{k+1} - \beta_k - [A^{n_k+2}(0 \oplus v_{k+1}),\ 1 \oplus 0])/(h(v_{k+1})2^{r_{k+1}}),$$

then $|\alpha_{k+1}| \le 2^{-k-1}$. (Here $\beta_k \oplus 0 = A^{n_k+2}(0 \oplus \sum_{j=1}^{k} x_j)$.) Finally,

$$x_{k+1} = 2^{-r_{k+1}} U^{n_{k+1} - n_k - p_{k+1} - 2}(v_{k+1}).$$

(Observe that x_{k+1} has been constructed so that

$$A^{n_{k+1} - n_k - p_{k+1} - 2}(0 \oplus x_{k+1}) = 2^{r_{k+1}} h(v_{k+1})\alpha_{k+1} \oplus v_{k+1}.)$$

To show that $0 \oplus x$ is hypercyclic, it is enough to verify that

$$\lim_{k\to\infty} \|(\beta_k \oplus z_k) - A^{n_k - p_k}(0 \oplus x)\| = 0.$$

On the other hand, since $g_0 g_1 \cdots g_{n_k} \leq 2^{-n_k}$, the unit circle does not intersect the point spectrum of G. Thus $\dim(\mathrm{Per}(A)) = 1$.

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Added in proof. In a paper in preparation by D. A. Herrero and the author, conditions for hypercyclicity of operator-valued weighted shifts are studied. In particular, hypercyclic weighted shifts are characterized in terms of their weights. As a consequence of this characterization, the hypercyclic bilateral weighted shifts whose adjoints are also hypercyclic are identified.

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