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INTEGRAL DOMAINS WITH FINITELY GENERATED GROUPS OF DIVISIBILITY

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ABSTRACT. Let D be an integral domain with integral closure $\overline{D}$. We show that the group of divisibility $G(D)$ of D is finitely generated if and only if $G(\overline{D})$ is finitely generated and $\overline{D}/[D:\overline{D}]$ is finite. We also show that $G(D)$ is finitely generated if and only if the monoid of finitely generated fractional ideals of D (under multiplication) is finitely generated.

Let D be an integral domain with quotient field K. Let K^* be the multiplicative group $K - \{0\}$ and $U(D)$ the group of units of D. The quotient group $G(D) = K^*/U(D)$, partially ordered by $aU(D) \leq bU(D) \Leftrightarrow a|b$ in D, is called the *group of divisibility* of D. Thus, the positive cone $G_+(D)$ is $D^*/U(D) = \{aU(D)|a \in D - \{0\}\}$. The group $G(D)$ is order isomorphic to the group $P(D)$ of nonzero principal fractional ideals of D (under multiplication) ordered by reverse inclusion, while $G_+(D)$ is order isomorphic to the submonoid $IP(D)$ of $P(D)$ consisting of nonzero principal integral ideals. However, for the most part, we will be interested only in the underlying group structure of $G(D)$.

The purpose of this paper is to characterize integral domains with finitely generated groups of divisibility. Theorem 3 states that $G(D)$ is finitely generated if and only if $G(\overline{D})$ is finitely generated and $\overline{D}/[D:\overline{D}]$ is finite, where $\overline{D}$ is the integral closure of D and $[D:\overline{D}] = \{d \in D|d\overline{D} \subseteq D\}$ is the conductor. Theorem 4 shows that an integral domain D with $G(D)$ finitely generated may be realized as a composite or pullback of a very special type. We also show (Theorem 5) that $G(D)$ is finitely generated if and only if the monoid $F^*(D)$ of nonzero finitely generated fractional ideals of D (under multiplication) is finitely generated.

The study of integral domains with finitely generated groups of divisibility was inaugurated by B. Glastad and J. L. Mott [2]. We collect in Theorem 0 results from [2] which will be used throughout this paper without further reference.

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Theorem 0 (B. Glastad and J. L. Mott). *Let D be an integral domain with $G(D)$ finitely generated.*

(1) *$\overline{D}$ is a Bézout domain with only finitely many prime ideals.*
(2) *$\overline{D}$ is a finitely generated D-module.*
(3) *$G(D) \cong G(\overline{D}) \oplus U(\overline{D})/U(D)$, where $G(\overline{D})$ is free and $U(\overline{D})/U(D)$ is finite.*
(4) *If P is a prime ideal of D with D/P infinite, then $D_P = \overline{D}_P$ is a valuation domain.*

Proof. (1) [2, Theorem 2.1].

(2) [2, Theorem 3.9].

(3) Consider the exact sequence $0 \to U(\overline{D})/U(D) \to G(D) \to G(\overline{D}) \to 0$. Then $G(\overline{D})$ is finitely generated and is also torsion-free since $\overline{D}$ is integrally closed. Hence $G(\overline{D})$ is free. Therefore, the exact sequence splits. By [2, Theorem 3.9], $U(\overline{D})/U(D)$ is finite.

(4) [2, Corollary 3.6]. □

Theorem 1. *Let D be an integral domain with $G(D)$ finitely generated. Then $\overline{D}/[D:\overline{D}]$ is finite.*

Proof. Since $\overline{D}$ is a finitely generated D-module, it suffices to show that $D/[D:\overline{D}]$ is finite. Also, since D is semi-quasilocal, it suffices to show that $D_M/[D:\overline{D}]_M$ is finite for each maximal ideal M of D containing $[D:\overline{D}]$. Because $\overline{D}$ is a finitely generated D-module, $[D:\overline{D}]_M = [D_M:\overline{D}_M]$. Thus, using the facts that $\overline{D}_M$ is the integral closure of D_M and that $G(D_M)$ is finitely generated, we have to show only that if (D, M) is a quasilocal domain with $G(D)$ finitely generated, then $D/[D:\overline{D}]$ is finite. If D/M is infinite, then $D = \overline{D}$, and the result follows. So we may assume that D/M is finite. Let $Q_1, \dots, Q_n$ be the maximal ideals of $\overline{D}$. Then $(1 + Q_1 \cap \dots \cap Q_n)/(1 + M)$, being isomorphic to a subgroup of $U(\overline{D})/U(D)$, is finite. Suppose that

$$(1 + Q_1 \cap \dots \cap Q_n)/(1 + M) = \{(1 + t_1)(1 + M), \dots, (1 + t_s)(1 + M)\}.$$

For $q \in Q_1 \cap \dots \cap Q_n$, $1 + q = (1 + t_i)(1 + m)$ for some i with $1 \le i \le s$ and $m \in M$. Thus $q = (1+m)t_i + m$, so $\overline{q} = (1+m)\overline{t}_i$ in $Q_1 \cap \dots \cap Q_n/M$. But $1+m$ is a unit, so $D\overline{q} = D\overline{t}_i$. Hence $Q_1 \cap \dots \cap Q_n/M$ has only finitely many cyclic D-submodules and hence has finite length as a D-module. Since D/M is finite, $Q_1 \cap \dots \cap Q_n/M$ is actually finite. Now $\overline{D}/Q_1 \cap \dots \cap Q_n \cong \overline{D}/Q_1 \times \dots \times \overline{D}/Q_n$ is finite, since each $\overline{D}/Q_i$ is a finitely generated D-module. Thus $\overline{D}/M$ is finite, and hence so is $\overline{D}/D$. Therefore $D/[D:\overline{D}] = D/\mathrm{ann}(\overline{D}/D)$ is finite. □

Thus $\overline{D}/[D:\overline{D}]$ is a finite principal ideal ring. Let $Q_1, \dots, Q_n$ be the maximal ideals of $\overline{D}$ containing $[D:\overline{D}]$. Then $[D:\overline{D}] = B_1 \cap \dots \cap B_n$, where B_i is Q_i-primary and $\overline{D}/[D:\overline{D}] \cong \overline{D}/B_1 \times \dots \times \overline{D}/B_n$, where $\overline{D}/B_i$ is a finite special principal ideal ring with finite residue field $\overline{D}/Q_i$. Since $\overline{D}/B_i$ is finite, either Q_i is principal, in which case $B_i = Q_i^{n_i}$ for some $n_i \ge 1$, or Q_i is not principal, and $B_i = Q_i$.

The following lemma, while probably well known, is stated for the convenience of the reader:

Lemma 2. *Let D be a semi-quasilocal ring and I an ideal of D. Let S be a subring of D/I and let $R = \pi^{-1}(S)$ where $\pi: D \to D/I$ is the natural map. Thus $U(D)/U(R) \cong U(D/I)/U(S)$.*

Proof. Let $Q_1, \ldots, Q_n$ be the maximal ideals of D. Now the natural map gives a group homomorphism $U(D) \to U(D/I)$. Moreover, this map is surjective. Let $x + I \in U(D/I)$. Since $(x, I) = D \not\subseteq Q_1 \cup \cdots \cup Q_n$, by [3, Theorem 124], there exists an $i \in I$ such that $x + i \notin Q_1 \cup \cdots \cup Q_n$. Thus $x + i \in U(D)$, and $x + I = (x + i) + I$. Hence the map $U(D) \to U(D/I)$ is surjective. So the map $U(D) \to U(D/I)/U(S)$ is surjective and it has kernel $U(R)$. Thus $U(D)/U(R) \cong U(D/I)/U(S)$. □

Theorem 3. *For an integral domain D, the following conditions are equivalent.*

(1) *$G(D)$ is finitely generated.*

(2) *$G(\overline{D})$ is finitely generated, and $\overline{D}/[D : \overline{D}]$ is finite.*

Proof. (1) $\Rightarrow$ (2). Suppose that $G(D)$ is finitely generated. Then $G(\overline{D})$, being a homomorphic image of $G(D)$, is also finitely generated. Theorem 1 gives the result that $\overline{D}/[D : \overline{D}]$ is finite.

(2) $\Rightarrow$ (1). Since $G(\overline{D})$ is finitely generated, $\overline{D}$ is a semi-quasilocal Bézout domain. Now $G(\overline{D})$ is a finitely generated free Abelian group, so the exact sequence $0 \to U(\overline{D})/U(D) \to G(D) \to G(\overline{D}) \to 0$ splits. Hence $G(D) \cong G(\overline{D}) \oplus U(\overline{D})/U(D)$. Moreover, by Lemma 2,

$$U(\overline{D})/U(D) \cong U(\overline{D}/[D : \overline{D}])/U(D/[D : \overline{D}])$$

is finite. So $G(D)$ is finitely generated. □

We next show that integral domains with finitely generated groups of divisibility may be realized as composites or pullbacks of a very special type.

Theorem 4. *Let D be an integral domain with $G(D)$ finitely generated. Then $\overline{D}$ is a Bézout domain with $G(\overline{D})$ a finitely generated free Abelian group, and $\overline{D}/[D : \overline{D}]$ is finite. Let $\pi: \overline{D} \to \overline{D}/[D : \overline{D}]$ be the natural map. Then $D = \pi^{-1}(D/[D : \overline{D}])$. Conversely, suppose that T is an integral domain with $G(T)$ finitely generated, and let I be an ideal of T with T/I finite. Let S be a subring of T/I. Then $D = \pi^{-1}(S)$, where $\pi: T \to T/I$ is the natural map, is an integral domain having the same quotient field as T, $I \subseteq [D : T]$, $\overline{D} = \overline{T}$, and $G(D)$ is finitely generated. If T is Bézout, $\overline{D} = T$ and $G(D) \cong G(T) \oplus U(T)/U(D) \cong G(T) \oplus U(T/I)/U(S)$.*

Proof. We have already proved the first part of Theorem 4. So suppose that T is an integral domain with $G(T)$ finitely generated. Then D is a subring of T containing I. Hence $I \subseteq [D : T]$, and D and T have the same quotient field. Since T/I is a finitely generated S-module, T is a finitely generated D-module. Hence $\overline{D} = \overline{T}$. Lemma 2 gives the result that $U(T)/U(D) \cong U(T/I)/U(S)$

is finite. From the exact sequence $0 \to U(T)/U(D) \to G(D) \to G(T) \to 0$ with the two outside groups finitely generated, we see that $G(D)$ is also finitely generated. If T is Bézout, then $\overline{D} = \overline{T} = T$, and $G(D) \cong G(\overline{D}) \oplus U(\overline{D})/U(D) \cong G(T) \oplus U(T/I)/U(S)$. $\square$

Let D be an integral domain with quotient field K, and let $F^*(D)$ be the monoid of nonzero finitely generated fractional ideals of D (under multiplication). Now if $F^*(D)$ is finitely generated, then the group $\mathrm{Inv}(D)$ of invertible ideals of D is also finitely generated, being generated by those generators for $F^*(D)$ which are invertible. Hence the subgroup $P(D)$ of nonzero principal fractional ideals of D is also finitely generated. Hence $G(D) \cong P(D)$ is finitely generated. (Since D is semi-quasilocal, we actually have $\mathrm{Inv}(D) = P(D)$.) We will show that the converse is true, but we first consider the more general situation where $\overline{D}$ is assumed only to be Bézout.

Suppose that D is an integral domain, with $\overline{D}$ being Bézout. The monoid homomorphism $F^*(D) \to P(\overline{D}) \cong G(\overline{D})$ given by $J \to J\overline{D}$ is surjective with "kernel" $B^*(D) = \{J \in F^*(D) | J\overline{D} = \overline{D}\}$. We have an exact monoid sequence $0 \to B^*(D) \to F^*(D) \to G(\overline{D}) \to 0$.

Suppose that, in addition to $\overline{D}$ being Bézout, the exact sequence $0 \to U(\overline{D})/U(D) \to G(D) \to G(\overline{D}) \to 0$ splits. (This is the case if $G(\overline{D})$ is finitely generated.) Let $g: G(\overline{D}) \to G(D)$ be a splitting. We may view g as a map from $P(\overline{D})$ to $P(D)$. In this context, the fact that g is a splitting amounts to saying that $g(\overline{D}x)\overline{D} = \overline{D}x$. Using g, we define the map $\theta: F^*(D) \to P(\overline{D}) \times B^*(D)$ by $\theta(J) = (J\overline{D}, (g(J\overline{D}))^{-1}J)$. We next show that θ is a monoid isomorphism and that $B^*(D)$ is finite when $G(D)$ is finitely generated.

Theorem 5. *Let D be an integral domain such that $\overline{D}$ is a Bézout domain and such that the exact sequence $0 \to U(\overline{D})/U(D) \to G(D) \to G(\overline{D}) \to 0$ splits. (This is the case if $G(D)$, or more generally $G(\overline{D})$, is finitely generated.) Then $F^*(D) \cong G(\overline{D}) \times B^*(D)$. If $G(D)$ is finitely generated, then $B^*(D)$ is finite. Thus $F^*(D)$ is finitely generated if and only if $G(D)$ is finitely generated.*

Proof. We identify $G(\overline{D})$ with $P(\overline{D})$. We show that the previously defined map θ is a monoid isomorphism. Since $J\overline{D} = g(J\overline{D})\overline{D}$, $g(J\overline{D})^{-1}J\overline{D} = \overline{D}$, so θ is well defined. The fact that θ is a monoid homomorphism easily follows from the fact that g is a homomorphism. If $\theta(J) = \theta(K)$, then $J\overline{D} = K\overline{D}$, so $g(J\overline{D}) = g(K\overline{D})$. Then $g(J\overline{D})^{-1}J = g(K\overline{D})^{-1}K$ implies that $J = K$. Finally, we show that θ is surjective. Let $(\overline{D}x, J) \in P(\overline{D}) \times B^*(D)$. So $J\overline{D} = \overline{D}$. Then

$$\begin{aligned} \theta(g(\overline{D}x)J) &= (g(\overline{D}x)J\overline{D}, (g(g(\overline{D}x)J\overline{D}))^{-1}g(\overline{D}x)J) \\ &= (g(\overline{D}x)\overline{D}, (g(g(\overline{D}x)\overline{D}))^{-1}g(\overline{D}x)J) \\ &= (\overline{D}x, (g(\overline{D}x))^{-1}g(\overline{D}x)J) \\ &= (\overline{D}x, DJ) = (\overline{D}x, J). \end{aligned}$$

Suppose that $G(D)$ is finitely generated. Let $J \in B^*(D)$. Then

$$[D : \overline{D}] = [D : \overline{D}]\overline{D} = [D : \overline{D}](J\overline{D}) = [D : \overline{D}]J \subseteq J \subseteq J\overline{D} = \overline{D}.$$

So $[D:\overline{D}] \subseteq J \subseteq \overline{D}$. But by Theorem 1, $\overline{D}/[D:\overline{D}]$ is finite. Hence $B^*(D)$ is finite. □

Corollary 6. *Let D be an integral domain with $G(D)$ finitely generated. Then there exists a finite set $\{A_1, \dots, A_n\}$ of finitely generated ideals of D such that every nonzero finitely generated fractional ideal of D has the form xA_i where $x \in K^*$ and $1 \le i \le n$. Hence there exists a natural number k such that every finitely generated fractional ideal of D can be generated by k elements.*

Suppose that D is an integral domain with $G(D)$ finitely generated. Then $B^*(D)$ is a finite monoid with identity D and zero $\overline{D}$. The group of units of $B^*(D)$ is $PB^*(D) = \{Dx | \overline{D}x = \overline{D}\} = \{Dx | x \in U(D)\}$, which is naturally isomorphic to $U(\overline{D})/U(D)$. Also, observe that a fractional ideal A of D is finitely generated if and only if $A\overline{D}$ is finitely generated. Certainly if A is finitely generated, then $A\overline{D}$ is finitely generated. Conversely, suppose that $A\overline{D}$ is finitely generated, say $A\overline{D} = \overline{D}x$. Then $x^{-1}A\overline{D} = \overline{D}$. Choose $x_1, \dots, x_n \in x^{-1}A$ with $(x_1, \dots, x_n)\overline{D} = \overline{D}$. So $(x_1, \dots, x_n) \in B^*(D)$. Then, as in the proof of Theorem 5, $[D:\overline{D}] \subseteq (x_1, \dots, x_n)D \subseteq x^{-1}A \subseteq \overline{D}$. Since $\overline{D}/[D:\overline{D}]$ is finite, $x^{-1}A/(x_1, \dots, x_n)D$ is finite. Hence $x^{-1}A$, and therefore A itself, is finitely generated.

If $G_+(D)$ is finitely generated, then certainly $G(D)$ is finitely generated. The converse is false. Now, $G_+(D) \cong IP(D)$ is finitely generated if and only if D has only a finite number of irreducible elements (up to units) and each nonzero nonunit element of D is a product of irreducible elements. Such domains, called *CK domains* in honor of Cohen and Kaplansky who first studied them, were the subject of [1]. It was shown [1, Theorem 4.5] that the following conditions on an integral domain D are equivalent:

(1) D is a CK domain,

(2) $G(D)$ is finitely generated and rank $G(D) = |\mathrm{Max}(D)|$ (we always have rank $G(D) \ge |\mathrm{Max}(D)|$),

(3) D is Noetherian, $G(D)$ is finitely generated, and $|\mathrm{Max}(D)| = |\mathrm{Max}(\overline{D})|$,

(4) $\overline{D}$ is a semilocal PID with $D/[D:\overline{D}]$ finite and $|\mathrm{Max}(D)| = |\mathrm{Max}(\overline{D})|$, and

(5) D is a one-dimensional semilocal domain such that for each nonprincipal maximal ideal M of D, D/M is finite and D_M is analytically irreducible.

In contrast, a Noetherian domain D has $G(D)$ finitely generated if and only if D is a one-dimensional semilocal domain such that for each nonprincipal maximal ideal M, D/M is finite and D_M is analytically unramified [1, Theorem 3.3]. It was observed [1, Corollary 3.6] that an integral domain D is a local CK domain if and only if $G(D) \cong \mathbb{Z} \oplus F$, where F is finite.

We end by examining integral domains D with $G(D) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus F$ where F is finite. First, suppose that D is an integral domain with $G(D) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus F$, where F is finite. Then $G(\overline{D}) \cong \mathbb{Z} \oplus \mathbb{Z}$, so $\overline{D}$ is a Bézout domain with at most two nonzero prime ideals. First, suppose that $\overline{D}$ has two maximal ideals. Then $\overline{D}$ is a semilocal PID with two maximal ideals. (In general, for a domain D with $G(D)$ finitely generated, $\overline{D}$, or equivalently D, is Noetherian if and only if rank $G(D) = |\mathrm{Max}(\overline{D})|$ [1, Corollary 3.5]). So either D has two maximal ideals, in which case D is a CK domain, or D is a one-dimensional local domain with finite residue field. Both cases may occur. For let K be a finite field and $T = K[X]_{(X)} \cap K[X]_{(X+1)}$ be a semilocal PID with two maximal ideals. Let $\pi: T \to K \oplus K$ be the natural map. If k is a proper subfield of K, then $D = \pi^{-1}(k \oplus k)$ is a CK domain with two maximal ideals that is not a PID, and $G(D) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus K^*/k^* \oplus K^*/k^*$. (Use Theorem 4 here and in the examples to follow.) If we take $K = \mathbb{Z}/2\mathbb{Z}$ and embed $K \to K \oplus K$ via the diagonal map, then $D = \pi^{-1}(K)$ is a local domain with $G(D) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Second, suppose that $\overline{D}$ has only one maximal ideal Q. So $\overline{D}$ is a valuation domain. If $\dim \overline{D} = 2$, then $\overline{D}$ must be a rank 2 discrete valuation domain. And either $D = \overline{D}$, or $\overline{D}/Q$ is finite and $D = \pi^{-1}(S)$, where S is a subring of the finite SPIR $\overline{D}/Q^n$. For example, if $k \subset K$ is a pair of finite fields, then for $D = k[[X]] + K((X))[[Y]]Y$, $\overline{D} = K[[X]] + K((X))[[Y]]Y$ is a rank 2 discrete valuation domain and $G(D) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus K^*/k^*$. Or, we may take $D = K[[X^2, X^3]] + K((X))[[Y]]Y$. Then $\overline{D} = K[[X]] + K((X))[[Y]]Y$ and $G(D) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus K$, where here K is the additive group of the field K. If $\dim \overline{D} = 1$, then $(\overline{D}, Q)$ is a one-dimensional valuation domain with $G(\overline{D}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Since $\overline{D}/[D : \overline{D}]$ must be finite, either $D = \overline{D}$ or $[D : \overline{D}] = Q$ and $\overline{D}/Q$ is finite. Thus $D = \pi^{-1}(k)$, where $\pi: \overline{D} \to \overline{D}/Q$ is the natural map and k is a subfield of the field $K = \overline{D}/Q$. Here $G(D) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus K^*/k^*$.

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