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AUTOMORPHISMS OF SL_2 OF IMAGINARY QUADRATIC INTEGERS

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ABSTRACT. We determine the outer automorphism groups of the two-dimensional special linear and projective special linear groups of the ring of integers in an imaginary quadratic field.

1. INTRODUCTION

The problem of computing the automorphism groups of $SL_n(R)$ and $PSL_n(R)$ for a commutative ring R is a problem with a long history (see, e.g. [10]). For $n \geq 3$ these automorphism groups have been completely characterized. When $n = 2$, the problem was solved for a large class of integral domains by M. Dull [4, 5]. If we assume R is the ring of integers in a number field, the only case not covered by Dull's theory is the ring of integers in the imaginary quadratic number field $\mathbf{Q}(\sqrt{-d})$, for $d \neq 1, 3$. In this paper we settle that case.

Let d be a squarefree positive integer, and let $\mathfrak{O} = \mathfrak{O}_{-d}$ denote the ring of integers in the imaginary quadratic field $\mathbf{Q}(\sqrt{-d})$. We first consider $PSL_2(\mathfrak{O}) = SL_2(\mathfrak{O})/\pm I$. The group $PSL_2(\mathfrak{O})$ acts on hyperbolic 3-space $\mathbf{H}^3$ by isometries. By the Mostow-Prasad rigidity theorem, any automorphism of $PSL_2(\mathfrak{O})$ is conjugation by an element of the full group of isometries of $\mathbf{H}^3$. The isometries of $\mathbf{H}^3$ are generated by $PSL(2, \mathbf{C})$ and complex conjugation, so we must determine which elements g of $PSL(2, \mathbf{C})$ give automorphisms of $PSL_2(\mathfrak{O})$. We show that any such element g sends the flag $\mathfrak{O} \cdot (1, 0) \subset \mathfrak{O} \oplus \mathfrak{O}$ to a flag L_g isomorphic to an element of order two in the class group of $\mathfrak{O}$; further, each element of order two in the class group determines a unique outer automorphism of $PSL_2(\mathfrak{O})$. The group $\mathscr{C}_2$ of elements of order two in the class group of $\mathfrak{O}$ is determined in [6]; it is isomorphic to $(\mathbf{Z}/2)^{t-1}$, where t is the number of distinct prime divisors of the discriminant of $\mathbf{Q}(\sqrt{-d})$. This leads to the following theorem.

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Theorem 5.1. *Let* $\mathscr{C}_2$ *denote the subgroup of elements of order two in the class group of* $\mathfrak{O}_{-d}$, *let* $\operatorname{ad} J(u)$ *denote conjugation by the matrix* $J(u) = \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$, *let* b *denote complex conjugation, and let* t *be the number of distinct prime divisors of the discriminant of* $\mathbf{Q}(\sqrt{-d})$. *If* $d \neq 1$ *then*

$$\operatorname{Out}(PSL_2(\mathfrak{O}_{-d})) \cong \mathscr{C}_2 \times \langle b \rangle \times \langle \operatorname{ad} J(-1) \rangle \cong (\mathbf{Z}/2)^{t+1};$$

if $d = 1$,

$$\operatorname{Out}(PSL_2(\mathfrak{O}_{-d})) \cong \langle b \rangle \times \langle \operatorname{ad} J(i) \rangle \cong (\mathbf{Z}/2)^2.$$

To determine the group of outer automorphisms of $SL_2(\mathfrak{O})$, we note that any outer automorphism of $SL_2(\mathfrak{O})$ is trivial on $\pm I$, hence gives an outer automorphism of $PSL_2(\mathfrak{O})$. We show that the resulting map $f: \operatorname{Out}(SL_2(\mathfrak{O})) \to \operatorname{Out}(PSL_2(\mathfrak{O}))$ splits, and we identify the kernel of f with $H^1(PSL_2(\mathfrak{O}); \mathbf{Z}/2)$, giving

Corollary 5.3.

$$\begin{aligned}\operatorname{Out}(SL_2(\mathfrak{O}_{-d})) &\cong \operatorname{Out}(PSL_2(\mathfrak{O}_{-d})) \ltimes H^1(PSL_2(\mathfrak{O}_{-d}); \mathbf{Z}/2) \\ &\cong (\mathbf{Z}/2)^{t+1} \ltimes H^1(PSL_2(\mathfrak{O}_{-d}); \mathbf{Z}/2),\end{aligned}$$

where t *is equal to the number of distinct prime divisors of the discriminant of* $\mathbf{Q}(\sqrt{-d})$.

For computations of $H^1(PSL_2(\mathfrak{O}_{-d}); \mathbf{Z}/2)$, see [11, 12]; in the euclidean cases, we have

$$\dim(H^1(PSL_2(\mathfrak{O}_{-d}); \mathbf{Z}/2)) = \begin{cases} 0 & \text{if } d = 3, \\ 1 & \text{if } d = 11, \\ 2 & \text{if } d = 1, 2, 7. \end{cases}$$

Our motivation for this theorem was to simplify the problem of finding a fundamental domain for the action of $PSL_2(\mathfrak{O})$ on $\mathbf{H}^3$, since automorphisms of $PSL_2(\mathfrak{O})$ correspond to symmetries of the fundamental domain (see [12]). If the class group of $\mathfrak{O}$ is an elementary abelian 2-group, the problem simplifies considerably.

2. Flags in $\mathfrak{O}_{-d} \oplus \mathfrak{O}_{-d}$

A *flag* L is a rank one torsion-free submodule of $\mathfrak{O} \oplus \mathfrak{O}$. Equivalently, a flag is a rank one submodule L of $\mathfrak{O} \oplus \mathfrak{O}$ that has a complement $L': L \oplus L' = \mathfrak{O} \oplus \mathfrak{O}$, or a flag is the intersection of a line in $\mathbf{Q}(\sqrt{-d}) \oplus \mathbf{Q}(\sqrt{-d})$ with the lattice $\mathfrak{O} \oplus \mathfrak{O} \subset \mathbf{Q}(\sqrt{-d}) \oplus \mathbf{Q}(\sqrt{-d})$. The group $PSL_2(\mathfrak{O})$ acts on the set of flags of $\mathfrak{O} \oplus \mathfrak{O}$. Let Γ_L denote the stabilizer of L. In this section we assign an $\mathfrak{O}$-module to Γ_L and show that if two stabilizers Γ_{L_1} and Γ_{L_2} are conjugate in $PSL(2, \mathbf{C})$, then the corresponding $\mathfrak{O}$-modules are isomorphic.

We remark first on complex lines: let l be a complex line in $\mathbf{C}^2$, and let P_l be the subgroup of $PSL(2, \mathbf{C})$ which fixes l. We can think of $PSL(2, \mathbf{C})$ as the group of complex automorphisms of $\mathbf{C}P^1$; if we let x_l denote the point of $\mathbf{C}P^1$ corresponding to l, then $\mathbf{C}P^1 - x_l$ has the structure of a complex affine line, and P_l is the group of complex affine transformations. Let $T_l \subset P_l$ consist of translations; T_l can be characterized as the maximal normal abelian subgroup of P_l. T_l is a complex abelian Lie group, of complex dimension 1.

Lemma 2.1. *Let l' be a line in $\mathbf{C}^2$ that is complementary to l. Then*

$$\mathrm{Hom}_{\mathbf{C}}(l', l) \cong T_l.$$

Proof. Given f in $\mathrm{Hom}_{\mathbf{C}}(l', l)$, define a translation

$$\varphi_f: (x + x') \mapsto (x + x' - f(x')),$$

where $x \in l$, $x' \in l'$. If we choose basis vectors for l and l', the map $f \mapsto \varphi_f$ takes the form $z \mapsto \left(\begin{smallmatrix} 1 & -z \\ 0 & 1 \end{smallmatrix}\right)$, where $z = f(1)$, showing that it is a holomorphic Lie group isomorphism. □

We now consider flags in $\mathfrak{O} \oplus \mathfrak{O}$. Let L be a flag with complement L'; let l and l' be the complex lines generated by L and L' respectively.

Proposition 2.2. $\mathrm{Hom}(L', L)$ *is isomorphic as a group to the subgroup $P_l \cap PSL_2(\mathfrak{O})$ of Γ_L.*

Proof. The isomorphism is given, as in Lemma 2.1, by associating to $f \in \mathrm{Hom}(L', L)$ the map $\varphi_f: (x + x') \mapsto (x + x' - f(x'))$, where $x \in L$ and $x' \in L'$. The map φ_f fixes L, has determinant 1, and takes $\mathfrak{O} \oplus \mathfrak{O}$ to itself; therefore, $\varphi_f \in P_l \cap PSL_2(\mathfrak{O})$.

The fact that $f \mapsto \varphi_f$ is an injective homomorphism follows from the complex case above. To see that it is surjective, we construct a section. Let $\pi: L \oplus L' \to L'$ be the natural projection, and let φ be an element of $P_l \cap PSL_2(\mathfrak{O})$. The map φ restricted to L is multiplication by ± 1, so we may lift φ to a unique element $\widehat{\varphi}$ in $SL_2(\mathfrak{O})$ with $\widehat{\varphi}|_L = \mathrm{id}$, and we have

$$\begin{array}{ccccccccc} 1 & \to & L & \to & \mathfrak{O} \oplus \mathfrak{O} & \xrightarrow{\pi} & L' & \to & 1 \\ & & \downarrow \mathrm{id} & & \downarrow \widehat{\varphi} & & & & \\ 1 & \to & L & \to & \mathfrak{O} \oplus \mathfrak{O} & \xrightarrow{\pi} & L' & \to & 1. \end{array}$$

The induced map on L' is an automorphism by the 5-lemma; since $\det \widehat{\varphi} = 1$, this map is also the identity, i.e., $\pi \circ \widehat{\varphi} = \pi$. Thus for any x' in L', $\pi(x' - \widehat{\varphi}(x')) = 0$, i.e., $x' = \widehat{\varphi}(x')$ is in $\ker(\pi) = L$. The section can now be defined by associating to φ the map $x' \mapsto (x' - \widehat{\varphi}(x'))$ in $\mathrm{Hom}(L', L)$. Note that if $d \neq 1, 3$ then $P_l \cap PSL_2(\mathfrak{O}) = \Gamma_L$ because any element of Γ_L restricts to an automorphism of L, i.e., multiplication by ± 1 (the only units in $\mathfrak{O}_{-d}$). □

Now consider two different flags L_1 and L_2, with complements L_1' and L_2', generating complex lines l_1, l_2, l_1', and l_2'.

Proposition 2.3. *Suppose Γ_{L_1} is conjugate to Γ_{L_2} by an element of $PSL_2(\mathbf{C})$. Then $\mathrm{Hom}(L_1', L_1)$ and $\mathrm{Hom}(L_2', L_2)$ are isomorphic as $\mathfrak{O}$-modules.*

Proof. If $g\Gamma_{L_1}g^{-1} = \Gamma_{L_2}$ for some g in $PSL_2(\mathbf{C})$, then $gl_1 = l_2$, and conjugation by g induces a complex Lie group isomorphism from P_{l_1} to P_{l_2}, which induces isomorphisms $P_{l_1} \cap PSL_2(\mathbf{C}) \to P_{l_2} \cap PSL_2(\mathbf{C})$ and $T_{l_1} \to T_{l_2}$, making the following diagram commute:

$$\begin{array}{ccc}
P_{l_1} \cap PSL_2(\mathbf{C}) & \xrightarrow{\cong} & P_{l_2} \cap PSL_2(\mathbf{C}) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}(L_1', L_1) & \xrightarrow{\gamma} & \mathrm{Hom}(L_2', L_2) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathbf{C}}(l_1', l_1) & & \mathrm{Hom}_{\mathbf{C}}(l_2', l_2) \\
\downarrow \cong & & \downarrow \cong \\
T_{l_1} & \xrightarrow{\cong} & T_{l_2}
\end{array}$$

The inclusions $\mathrm{Hom}(L_i', L_i) \to \mathrm{Hom}_{\mathbf{C}}(l_i', l_i)$ are $\mathfrak{O}$-module maps, so γ is an $\mathfrak{O}$-module isomorphism. □

3. Flags and the class group

Every projective module L over a dedekind domain Λ is isomorphic to $\mathfrak{A} \oplus \Lambda^n$ for some ideal $\mathfrak{A}$ of Λ and some n [8]. We let [L] denote the element of the class group of Λ corresponding to $\mathfrak{A}$.

Proposition 3.1. *Let L be a flag in $\mathfrak{O} \oplus \mathfrak{O}$, with complement L'. Then*

$$[\mathrm{Hom}(L', L)] = 2[L].$$

Proof. $\mathrm{Hom}(L', L) \cong L' \otimes_{\mathfrak{O}} L$, so

$$(*) \qquad [\mathrm{Hom}(L', L)] = [L'^* \otimes_{\mathfrak{O}} L] = [L'^*] + [L],$$

because tensor product of modules is addition in the class group.

For any ideal $\mathfrak{A} \subseteq \mathfrak{O}$, we can identify $\mathrm{Hom}(\mathfrak{A}, \mathfrak{A})$ with a subring of $\mathbf{Q}(\sqrt{-d})$. If $x \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{A})$, then the norm of x is equal to the index $[\mathfrak{A}: x\mathfrak{A}] \in \mathbf{Z}$; thus $x \in \mathfrak{O}$. Since $\mathfrak{O} \subseteq \mathrm{Hom}(\mathfrak{A}, \mathfrak{A}) \subseteq \mathfrak{O}$, we have $\mathrm{Hom}(\mathfrak{A}, \mathfrak{A}) = \mathfrak{O}$. In particular, $0 = [\mathrm{Hom}(L', L)] = [L'^* \otimes_{\mathfrak{O}} L] = [L'^*] + [L]$, so $[L'^*] = -[L]$. Substituting into $(*)$, we get

$$[\mathrm{Hom}(L', L)] = [L] - [L'].$$

Since $L \oplus L' = \mathfrak{O} \oplus \mathfrak{O}$, we have $[L'] = -[L]$, so $[\mathrm{Hom}(L', L)] = 2[L]$. □

Corollary 3.2. *Let L and M be flags in $\mathfrak{O} \oplus \mathfrak{O}$. If Γ_L is conjugate to Γ_M in $PSL_2(\mathbf{C})$, then $2[L] = 2[M]$.*

Proof. This is immediate from Propositions 2.3 and 3.1. □

We have the following converse to Corollary 3.2.

Proposition 3.3. *Let L and M be two flags in $\mathfrak{O} \oplus \mathfrak{O}$ with $2[L] = 2[M]$. Then Γ_L is conjugate to Γ_M by an element of $GL_2(\mathbf{Q}(\sqrt{-d}))$, and conjugation by this element gives an automorphism of $SL_2(\mathfrak{O})$.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be ideals in $\mathfrak{O}$, with $[\mathfrak{A}] = -[L]$ and $[\mathfrak{B}] = -[M]$. Since $2[\mathfrak{A}] = 2[\mathfrak{B}]$, we can choose an isomorphism $g: \mathfrak{A} \oplus \mathfrak{A} \to \mathfrak{B} \oplus \mathfrak{B}$ with $g \in GL_2(\mathbf{Q}(\sqrt{-d}))$.

Note first that conjugation by g gives an automorphism of $SL_2(\mathfrak{O})$: for any $h \in SL_2(\mathfrak{O})$, $g^{-1}hg: \mathfrak{A} \oplus \mathfrak{A} \to \mathfrak{A} \oplus \mathfrak{A}$ is an isomorphism. Thus $g^{-1}hg$ gives an automorphism of $\mathfrak{A}^{-1}(\mathfrak{A} \oplus \mathfrak{A}) = \mathfrak{O} \oplus \mathfrak{O}$ and has determinant 1, so is in $SL_2(\mathfrak{O})$.

Since g is an isomorphism, there are generators a_1, a_2 of $\mathfrak{A}$ and b_1, b_2 of $\mathfrak{B}$, with $g(a_1, a_2) = (b_1, b_2)$, i.e., g takes the flag $L' = \mathbf{Q}(\sqrt{-d})(a_1, a_2) \cap \mathfrak{O} \oplus \mathfrak{A} = \mathfrak{A}^{-1}(a_1, a_2)$ to $M' = \mathfrak{B}^{-1}(b_1, b_2)$. Since $[L'] = [\mathfrak{A}^{-1}] = -[A] = [L]$ and $[M'] = [M]$, there are matrices h and k in $SL_2(\mathfrak{O})$ with $hL = L'$ and $kM = M'$. Then $k^{-1}gh$ is in $GL_2(\mathbf{Q}(\sqrt{-d}))$, takes L to M and gives an automorphism of $SL_2(\mathfrak{O})$. □

4. Flags and cusps

In the next section it will be more convenient to use the notion of cusps in $\mathbf{C} \cup \{\infty\}$ in place of flags in $\mathfrak{O} \oplus \mathfrak{O}$. The correspondence is as follows:

A point λ in the extended complex plane is called a *cusp* if $\lambda \in \mathbf{Q}(\sqrt{-d}) \cup \{\infty\}$. We can identify the set of cusps with the set of flags in $\mathfrak{O} \oplus \mathfrak{O}$ by sending $\lambda = a/b$ $(a, b \in \mathfrak{O})$ to the flag $L_\lambda = \mathbf{Q}(\sqrt{-d})(a, b) \cap \mathfrak{O} \oplus \mathfrak{O} = \langle a, b\rangle^{-1}(a, b)$, where $\langle a, b\rangle$ is the ideal in $\mathfrak{O}$ generated by a and b [7]. The *class* of a cusp λ, denoted $[\lambda]$, is the class of the module L_λ, i.e., $[\lambda] = -[\langle a, b\rangle]$. The action of $PSL_2(\mathfrak{O})$ on the set of cusps is given by linear fractional transformations: $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \cdot \frac{a}{b} = \frac{xa+yb}{za+wb}$. If we let Γ_λ denote the stabilizer in $PSL_2(\mathfrak{O})$ of λ, we can restate Corollary 3.2 and Proposition 3.3 as follows:

Proposition 4.1. *Let λ and μ be cusps. If Γ_λ is conjugate to Γ_μ in $PSL_2(\mathbf{C})$, then $2[\lambda] = 2[\mu]$.*

Proposition 4.2. *Let λ and μ be cusps with $2[\lambda] = 2[\mu]$. Then Γ_λ is conjugate to Γ_μ by an element of $GL_2(\mathbf{Q}(\sqrt{-d}))$ which gives an automorphism of $SL_2(\mathfrak{O})$.*

5. Automorphisms of $SL_2(\mathfrak{O}_{-d})$

In this section we prove the following theorem:

Theorem 5.1. *Let $\mathscr{C}_2$ denote the subgroup of elements of order two in the class group, let $\operatorname{ad} J(u)$ denote conjugation by the matrix $J(u) = \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$, let b denote complex conjugation, and let t be the number of distinct prime divisors of the discriminant of $\mathbf{Q}(\sqrt{-d})$. If $d \neq 1$ then*

$$\operatorname{Out}(PSL_2(\mathfrak{O}_{-d})) \cong \mathscr{C}_2 \times \langle b\rangle \times \langle \operatorname{ad} J(-1)\rangle \cong (\mathbf{Z}/2)^{t+1};$$

if $d = 1$,

$$\operatorname{Out}(PSL_2(\mathfrak{O}_{-d})) \cong \langle b \rangle \times \langle \operatorname{ad} J(i) \rangle \cong (\mathbf{Z}/2)^2.$$

Proof. By the theorem of Mostow and Prasad [9], any automorphism of $PSL_2(\mathfrak{O})$ is given by conjugation by an isometry of hyperbolic 3-space $\mathbf{H}^3$, the symmetric space for $SL_2(\mathbf{C})$. If the isometry preserves an orientation of $\mathbf{H}^3$, we say the automorphism is orientation preserving. The group of orientation-preserving isometries of $\mathbf{H}^3$ is isomorphic to $PSL_2(\mathbf{C})$.

Let γ be an element of $PSL_2(\mathbf{C})$ which gives an automorphism of $PSL_2(\mathfrak{O})$. We first show that conjugation by γ is the same as conjugation by an element of $GL_2(\mathbf{Q}(\sqrt{-d}))$.

Claim. $\gamma \cdot \infty \in \mathbf{Q}(\sqrt{-d}) \cup \infty$.

Proof. Let $\gamma = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$. Because γ gives an automorphism of $PSL_2(\mathfrak{O})$,

$$\gamma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \gamma^{-1} = \begin{pmatrix} 1 - zx & x^2 \\ -z^2 & 1 + zx \end{pmatrix}$$

has entries in $\mathfrak{O}$; because xz and x^2 are in $\mathfrak{O}$, $\gamma \cdot \infty = x/z$ is in $\mathbf{Q}(\sqrt{-d}) \cup \{\infty\}$.

Because $\Gamma_{x/z} = \gamma^{-1}\Gamma_\infty\gamma$, we have $2[x/z] = 2[\infty] = 0$ by Proposition 4.1. By Proposition 4.2 we can choose $\gamma_1 \in GL_2(\mathbf{Q}(\sqrt{-d}))$ sending x/z back to ∞, such that conjugation by γ_1 gives an automorphism of $PSL_2(\mathfrak{O})$.

Now $\gamma_1\gamma$ stabilizes ∞, and hence has the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$; because $\gamma_1\gamma$ gives an automorphism of $PSL_2(\mathfrak{O})$, so does $h = \frac{1}{a}\gamma_1\gamma = \left(\begin{smallmatrix} 1 & b' \\ 0 & c' \end{smallmatrix}\right)$. In particular,

$$h \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} h^{-1} = \begin{pmatrix} -b' & * \\ -c' & -b' \end{pmatrix}$$

has entries in $\mathfrak{O}$, so b' and c' are in $\mathfrak{O}$. Thus $\gamma/a = \gamma_1^{-1}\left(\begin{smallmatrix} 1 & b' \\ 0 & c' \end{smallmatrix}\right)$ is in $GL_2(\mathbf{Q}(\sqrt{-d}))$, so conjugation by γ is equivalent to conjugation by an element of $GL_2(\mathbf{Q}(\sqrt{-d}))$.

We now consider this element of $GL_2(\mathbf{Q}(\sqrt{-d}))$. After clearing denominators, we may assume it has the form $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with a, b, c, and d in $\mathfrak{O}$. The following lemma gives restrictions on g.

Lemma 5.2. *If conjugation by* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbf{Q}(\sqrt{-d}))$ *gives an automorphism of* $PSL_2(\mathfrak{O})$, *then the product of any two entries of* g *is in the fractional ideal generated by* $\det g$.

Proof.

$$g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g^{-1} - g \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} g^{-1} = g \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} g^{-1} = \frac{1}{\det g} \begin{pmatrix} -bc & ab \\ -cd & ad \end{pmatrix};$$

$$g \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g^{-1} - g \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g^{-1} = g \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g^{-1} = \frac{1}{\det g} \begin{pmatrix} -ac & a^2 \\ c^2 & ac \end{pmatrix};$$

$$g\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}g^{-1} - g\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}g^{-1} = g\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}g^{-1} = \frac{1}{\det g}\begin{pmatrix}bd & -b^2\\ d^2 & -bd\end{pmatrix}.$$

Because all the matrices on the right must be in $M_2\mathfrak{O}$, and the product of any two entries of g occurs in at least one of these matrices, the lemma is proved. □

Using the lemma, it is easy to check that the matrix $g = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ gives an isomorphism of $\mathfrak{O} \oplus \mathfrak{O}$ onto $\langle a, b\rangle \oplus \langle c, d\rangle$, (where $\langle a, b\rangle$ denotes the ideal generated by a and b, etc.).

Claim. $\langle a, b\rangle = \langle c, d\rangle$.

Proof. Let $L = \langle a, b\rangle^{-1}(-b, a)$ and $L' = \langle c, d\rangle^{-1}(-d, c)$. Another application of the lemma shows that $L' \oplus L = \mathfrak{O} \oplus \mathfrak{O}$ (e.g. $(1, 0) = a(d, -c)/\det g + c(-b, a)/\det g$). The matrix $h = g^{-1}\left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)g$ is in $SL_2(\mathfrak{O})$, $h: L' \oplus L \xrightarrow{\cong} L \oplus L'$ and $h^2 = -I$; since $h(L') \subseteq L$ and $h(L) \subseteq L'$, we have $L' \cong L$, i.e., $\langle a, b\rangle \cong \langle c, d\rangle$. Thus $\langle a, b\rangle = x\langle c, d\rangle$ for some $x \in \mathbf{Q}(\sqrt{-d})$. Because $g: \mathfrak{O} \oplus \mathfrak{O} \to \langle a, b\rangle \oplus x\langle a, b\rangle$ is an isomorphism, we must have $c = xc'$ and $d = xd'$, where $g' = \left(\begin{smallmatrix} a & b\\ c' & d'\end{smallmatrix}\right): \mathfrak{O} \oplus \mathfrak{O} \to \langle a, b\rangle \oplus \langle a, b\rangle$ is an isomorphism and hence gives an automorphism of $SL_2(\mathfrak{O})$. Let $A = g'^{-1}\left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)g'$. Then $gAg^{-1} = \left(\begin{smallmatrix} 0 & x^{-1}\\ x^{-1} & 0\end{smallmatrix}\right)$; because this has entries in $\mathfrak{O}$, x is a unit in $\mathfrak{O}$, hence $\langle a, b\rangle = \langle c, d\rangle$. □

Remark. According to J. E. Cremona [3], this computation can be found in Bianchi's paper [1].

Thus $\langle a, b\rangle$ has order two in the class group. If $\langle a, b\rangle$ is principal, conjugation by g is equivalent to conjugation by an element of $GL_2\mathfrak{O}$, that is, the automorphism is either inner, or inner composed with conjugation by $J(u) = \left(\begin{smallmatrix} u & 0\\ 0 & 1\end{smallmatrix}\right)$ for appropriate u; thus we may assume $\langle a, b\rangle$ is a nontrivial element of $\mathscr{C}_2$.

We will now define an injective homomorphism from $\mathscr{C}_2$ to $\mathrm{Out}(PSL_2(\mathfrak{O}))$. Choose a set of generators $\alpha_1, \dots, \alpha_{t-1}$ for $\mathscr{C}_2$. For each α_i, choose an ideal $\mathfrak{A}_i$ representing α_i and an isomorphism $g_i: \mathfrak{O} \oplus \mathfrak{O} \to \mathfrak{A}_i \oplus \mathfrak{A}_i$. Since $\mathfrak{A}_i^2 = \langle N(\mathfrak{A}_i)\rangle$ (see, e.g. [2]), $g_i^2: \mathfrak{O} \oplus \mathfrak{O} \xrightarrow{\cong} \langle N(\mathfrak{A}_i)\rangle \oplus \langle N(\mathfrak{A}_i)\rangle$ has determinant $u^2 N(\mathfrak{A}_i)^2 \in \mathbf{Z}$; thus $\det g_i = uN(\mathfrak{A}_i)$ with $u \in \mathfrak{O}^*$, so we may choose g_i to have determinant $N(\mathfrak{A}_i) > 0$.

The map from $\mathscr{C}_2$ to $\mathrm{Out}(PSL_2(\mathfrak{O}))$ is defined by sending each generator α_i to the automorphism γ_i given by conjugation by g_i. We check the relations $\alpha_i^2 = 1$ and $(\alpha_i\alpha_j)^2 = 1$ to see that this map is a homomorphism; (e.g. γ_i^2 is equivalent to conjugation by $g_i^2/N(\mathfrak{A}_i)$, which is in $SL_2(\mathfrak{O})$, so γ_i^2 is inner).

To see that the map is injective, take $\alpha = \alpha_{i_1} \cdots \alpha_{i_k}$ in its kernel. Then conjugation by $g_{i_k} \cdots g_{i_1}: \mathfrak{O} \oplus \mathfrak{O} \to \mathfrak{A}_{i_k} \cdots \mathfrak{A}_{i_1} \oplus \mathfrak{A}_{i_k} \cdots \mathfrak{A}_{i_1}$ is inner, so $\mathfrak{A}_{i_k} \cdots \mathfrak{A}_{i_1}$ is principle, i.e., $\alpha = 1$.

Now let η be the image of $[\langle a, b\rangle]$ in $\mathrm{Out}(PSL_2(\mathfrak{O}))$; thus η is given by conjugation by an isomorphism $h: \mathfrak{O} \oplus \mathfrak{O} \to \mathfrak{A} \oplus \mathfrak{A}$ with $[\mathfrak{A}] = [\langle a, b\rangle]$, i.e.,

$\mathfrak{A} = x\langle a, b\rangle$ for some $x \in \mathbf{Q}(\sqrt{-d})$. If $\det xg = u \det h$, conjugation by g is conjugation by η composed with conjugation by $J(u)$.

We have shown that $\mathrm{Out}(PSL_2(\mathfrak{O}))$ is generated by $\mathscr{C}_2$, conjugation by $J(u)$, and complex conjugation. It remains only to check that these generators commute. Complex conjugation commutes with conjugation by $J(-1)$ in $\mathrm{Aut}(SL_2(\mathfrak{O}))$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right): \mathfrak{O} \oplus \mathfrak{O} \xrightarrow{\cong} \langle a, b\rangle \oplus \langle a, b\rangle$, then

$$J(-1)gJ^{-1}(-1)g^{-1} = \frac{1}{\det g}\begin{pmatrix} ad+bc & -2ab \\ -2cd & ad+bc \end{pmatrix},$$

which is in $SL_2(\mathfrak{O})$ by Lemma 5.2; thus conjugation by $J(-1)$ commutes with elements of $\mathscr{C}_2$ up to inner automorphism. Since $\mathfrak{A}^2$ is principal, $\mathfrak{A} = \overline{A}$ (see, e.g. [2]); thus $\langle a, b\rangle = \langle c, d\rangle = \langle \overline{a}, \overline{b}\rangle = \langle \overline{c}, \overline{d}\rangle$, and all the entries of

$$g\overline{g}^{-1} = \frac{1}{\det g}\begin{pmatrix} a\overline{d} - b\overline{c} & -a\overline{b} + \overline{a}b \\ c\overline{d} - d\overline{c} & c\overline{b} - \overline{a}d \end{pmatrix}$$

are in $\langle a, b\rangle^2 = \langle \det g\rangle$. Thus $g\overline{g}^{-1}$ is in $SL_2(\mathfrak{O})$ and b commutes with elements of $\mathscr{C}_2$ up to inner automorphism. If $d = 1$, conjugation by $J(-1)$ is inner.

Corollary 5.3. $\mathrm{Out}(SL_2(\mathfrak{O}_{-d})) \cong (\mathbf{Z}/2)^{t+1} \ltimes H^1(PSL_2(\mathfrak{O}_{-d}); \mathbf{Z}/2)$, *where* t *is the number of distinct prime divisors in the discriminant of* $\mathbf{Q}(\sqrt{-d})$.

Proof. There is a map $f: \mathrm{Out}(SL_2(\mathfrak{O})) \to \mathrm{Out}(PSL_2(\mathfrak{O}))$, since any automorphism of $SL_2(\mathfrak{O})$ is trivial on the center $\pm I$. This map splits: an automorphism of $PSL_2(\mathfrak{O})$ is given by complex conjugation or conjugation by g for some $g \in PSL_2(\mathbf{C})$; since conjugation by g is the same as conjugation by $-g$, any such automorphism gives an automorphism of $SL_2(\mathfrak{O})$. The kernel of f is the subgroup of automorphisms α of $SL_2(\mathfrak{O})$ which fix $PSL_2(\mathfrak{O})$, i.e., $\alpha A = \pm A$ for all A in $SL_2(\mathfrak{O})$. Any such α gives a homomorphism $x(\alpha): SL_2(\mathfrak{O}) \to \{\pm I\} = \mathbf{Z}/2$, sending A to $\alpha(A)A^{-1}$. This homomorphism factors through $PSL_2(\mathfrak{O})$ because it is trivial on the center of $SL_2(\mathfrak{O})$, and it is easy to see that the map $\alpha \mapsto x(\alpha)$ gives an isomorphism from the kernel of f to $\mathrm{Hom}(PSL_2(\mathfrak{O}), \mathbf{Z}/2) \cong H^1(PSL_2(\mathfrak{O}); \mathbf{Z}/2)$. Because $\mathrm{Out}(PSL_2(\mathfrak{O})) \cong (\mathbf{Z}/2)^{t+1}$ by the theorem, the split short exact sequence:

$$1 \to H^1(PSL_2(\mathfrak{O}); \mathbf{Z}/2) \to \mathrm{Out}(SL_2(\mathfrak{O})) \to (\mathbf{Z}/2)^{t+1} \to 0$$

gives the corollary. □

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