PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 3, July 1991

# THE DERIVATIVE OF THE EXPONENTIAL MAP

CARL HERZ

(Communicated by Jonathan M. Rosenberg)

ABSTRACT. We give a quick analytic derivation of the formula for the derivative of the exponential of a vector field on a manifold.

Our object is to give a quick proof of a result usually obtained only for analytic manifolds using combinatorial manipulation with power series.

Let $\mathscr{M}$ be a $C^{(\infty)}$-manifold. Put $\mathrm{Vec}(\mathscr{M})$ for the space of $C^{(\infty)}$-vector fields on $\mathscr{M}$. Given $X \in \mathrm{Vec}(\mathscr{M})$ and $p \in \mathscr{M}$ we write $\exp(tX)p$ for the point on the manifold corresponding to the flow of $X$ at time $t$ that passes through $p$ at time $t = 0$. Thus one has

$$(d/dt)f \circ \exp(tX)|_{t=0} = f' \circ X$$

everywhere on $\mathscr{M}$, where $f'$ is the derivative of the mapping $f: \mathscr{M} \to \mathbb{R}$. If $\exp(tX)$ is globally defined then

$$(d/dt)f \circ \exp(tX) = f' \circ X \circ \exp(tX) = f' \circ T(\exp(tX)) \circ X.$$

Here we use the notation $T(S): T(\mathscr{M}) \to T(\mathscr{M})$ for the functorial map of the tangent bundle corresponding to a $C^{(\infty)}$-map $S: \mathscr{M} \to \mathscr{M}$.

Let $S$ be an automorphism of $\mathscr{M}$. Given $V \in \mathrm{Vec}(\mathscr{M})$ we get a new vector field

$$(\mathrm{Ad}\, S)\, V \overset{\text{def}}{=} T(S) \circ V \circ S^{-1}.$$

(1) **Proposition.** *Suppose $X$, $V \in \mathrm{Vec}(\mathscr{M})$ and that $\exp X$ is globally defined. Then, for any test function $f \in C^{(\infty)}(\mathscr{M})$ we have*

$$(d/dt)f \circ (\exp(X + tV)) \circ \exp(-X)|_{t=0} = f' \circ \int_0^1 \mathrm{Ad}(\exp sX)\, V\, ds.$$

*Remark.* $X$ and $V$ are fixed in the statement. Thus, for each $p \in \mathscr{M}$, $s \mapsto \mathrm{Ad}(\exp sX)\, V(p)$ gives a continuous map $[0, 1] \to T_p(\mathscr{M})$. $f'$ is a linear functional on the finite-dimensional vector space $T_p(\mathscr{M})$. Thus one has an

Received by the editors June 6, 1990 and, in revised form, November 21, 1990.
1991 *Mathematics Subject Classification*. Primary 58A30, 22E30.
*Key words and phrases.* Vector field, exponential map.
Research supported by the Natural Science and Engineering Research Council of Canada and les Fonds FCAC du Québec.

ordinary vector-valued integral that commutes with the action of linear functionals.

*Proof*. For $0 \leq s \leq 1$ and small $t$ put

$$F(s, t) = f \circ \exp\big(s(X + tV)\big) \circ \exp(-sX).$$

It is clear that one gets a vector field $W_s$ defined, for $s > 0$, by

$$f' \circ sW_s = (\partial/\partial t)F(s, t)|_{t=0}. \tag{2}$$

On the other hand we have

$$(\partial/\partial s)F(s, t) = f' \circ T\Big(\exp\big(s(X + tV)\big)\Big) \circ tV \circ \exp(-sX). \tag{3}$$

If we differentiate the line above with respect to $t$ at $t = 0$, then, using the chain rule, all terms drop out except the differentiation of $tV$, and we get

$$f' \circ \mathrm{Ad}(\exp sX)\, V.$$

Comparing the derivative of (2) with respect to $s$ and the derivative of (3) with respect to $t$ at $t = 0$, one has

$$(d/ds)sf' \circ W_s = f' \circ \mathrm{Ad}(\exp sX)\, V,$$

and integrating from 0 to 1 gives the conclusion.

*Remark*. It is not necessary that $\exp X$ be globally defined. The formula of Proposition (1) holds at any point $p \in \mathscr{M}$ such that the flow of $X$ with initial condition at $p$ is defined in a neighborhood of the time interval $[-1, 0]$; this is to say that $\exp(-X)p$ is defined.

The commutator of vector fields defined on test functions by

$$f' \circ [V, W] = (f' \circ V)' \circ W - (f' \circ W)' \circ V.$$

A straightforward calculation gives

$$(d/ds)\,\mathrm{Ad}(\exp sX)\, V = \mathrm{Ad}(\exp sX)\,[X, V] = [X, \mathrm{Ad}(\exp sX)\, V]. \tag{4}$$

Thus $\mathrm{ad}\, X$ defined on $\mathrm{V\small{EC}}(\mathscr{M})$ by $\mathrm{ad}\, X\, V = [X, V]$ is the infinitesimal generator of the one-parameter group of operators $s \mapsto \mathrm{Ad}(\exp sX)$. We may therefore write

$$\mathrm{Ad}(\exp sX) \equiv \exp(s\, \mathrm{ad}\, X)$$

with the understanding that the right-hand side is to be interpreted in the general case as the left-hand side. On an analytic manifold one may expand

$$\exp(s\, \mathrm{ad}\, X)V = \sum_{k=0}^{\infty}(1/k!)s^k(\mathrm{ad}\, X)^k V. \tag{5}$$

This formula remains valid if there exists a finite-dimensional Lie algebra $\mathfrak{g} \subset \mathrm{V\small{EC}}(\mathscr{M})$ such that $X, V \in \mathfrak{g}$. Let us define

$$\psi(u) = \int_0^1 e^{su}\, ds = u^{-1}(e^u - 1).$$

The formula of (1) may be rewritten to give

(6) **Theorem.** *If* $X$, $V \in \mathrm{Vec}(\mathscr{M})$ *then at any point* $p \in \mathscr{M}$ *for which* $\exp(-X)p$ *is defined we have (composition with test functions being understood)*

$$(d/dt)\exp(X + tV) \circ \exp(-X)|_{t=0} = \psi(\operatorname{ad} X)\, V. \tag{7}$$

This formula with $\psi$ expanded as a power series gives the usual expression, [1, Chapter II, Theorem 1.7]. (The difference in sign results from the fact that we have differentiation acting on the right instead of the left.) In the context of Lie groups, there is a somewhat longer proof in the same spirit as ours in [2, Theorem 2.14.2]. Note that the power series expansion is valid not only on analytic manifolds but also if $\mathscr{M}$ is merely a $C^{(2)}$-manifold provided that $X$ and $V$ are $C^{(2)}$-vector fields generating a finite-dimensional Lie algebra.

In the application to Lie groups $\mathbf{G}$ one considers the left multiplication operator $L_S$ and the right multiplication operator $R_S$ corresponding to an element $S \in \mathbf{G}$. The Lie algebra $\mathfrak{g}$ is identified as $T_I(\mathbf{G})$, and one associates to $X \in \mathfrak{g}$ the infinitesimal left translation (right-invariant vector field) $\Lambda_X$ by defining $\Lambda_X(S) = T(R_S)X$. The bracket in the Lie algebra is given by $[X, Y] = [\Lambda_X, \Lambda_Y](I)$. Since left- and right-multiplication commute, we have $(\operatorname{Ad} L_S)\Lambda_X = \Lambda_Y$ where $Y = (\operatorname{Ad} S)X$ in the usual sense of the adjoint action of $\mathbf{G}$ on $\mathfrak{g}$. Thus (7) extends the standard formula for Lie groups to general vector fields on a manifold.

## References


1. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, 1978.
2. V.S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Prentice-Hall, Englewood Cliffs, NJ, 1974.


Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, Montreal H3A 2K6, Canada