

## THE DERIVATIVE OF THE EXPONENTIAL MAP

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**ABSTRACT.** We give a quick analytic derivation of the formula for the derivative of the exponential of a vector field on a manifold.

Our object is to give a quick proof of a result usually obtained only for analytic manifolds using combinatorial manipulation with power series.

Let  $\mathcal{M}$  be a  $C^{(\infty)}$ -manifold. Put  $\text{VEC}(\mathcal{M})$  for the space of  $C^{(\infty)}$ -vector fields on  $\mathcal{M}$ . Given  $X \in \text{VEC}(\mathcal{M})$  and  $p \in \mathcal{M}$  we write  $\exp(tX)p$  for the point on the manifold corresponding to the flow of  $X$  at time  $t$  that passes through  $p$  at time  $t = 0$ . Thus one has

$$(d/dt)f \circ \exp(tX)|_{t=0} = f' \circ X$$

everywhere on  $\mathcal{M}$ , where  $f'$  is the derivative of the mapping  $f: \mathcal{M} \rightarrow \mathbb{R}$ . If  $\exp(tX)$  is globally defined then

$$(d/dt)f \circ \exp(tX) = f' \circ X \circ \exp(tX) = f' \circ T(\exp(tX)) \circ X.$$

Here we use the notation  $T(S): T(\mathcal{M}) \rightarrow T(\mathcal{M})$  for the functorial map of the tangent bundle corresponding to a  $C^{(\infty)}$ -map  $S: \mathcal{M} \rightarrow \mathcal{M}$ .

Let  $S$  be an automorphism of  $\mathcal{M}$ . Given  $V \in \text{VEC}(\mathcal{M})$  we get a new vector field

$$(\text{Ad } S)V \stackrel{\text{def}}{=} T(S) \circ V \circ S^{-1}.$$

(1) **Proposition.** *Suppose  $X, V \in \text{VEC}(\mathcal{M})$  and that  $\exp X$  is globally defined. Then, for any test function  $f \in C^{(\infty)}(\mathcal{M})$  we have*

$$(d/dt)f \circ (\exp(X + tV)) \circ \exp(-X)|_{t=0} = f' \circ \int_0^1 \text{Ad}(\exp sX) V ds.$$

*Remark.*  $X$  and  $V$  are fixed in the statement. Thus, for each  $p \in \mathcal{M}$ ,  $s \mapsto \text{Ad}(\exp sX)V(p)$  gives a continuous map  $[0, 1] \rightarrow T_p(\mathcal{M})$ .  $f'$  is a linear functional on the finite-dimensional vector space  $T_p(\mathcal{M})$ . Thus one has an

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ordinary vector-valued integral that commutes with the action of linear functionals.

*Proof.* For  $0 \leq s \leq 1$  and small  $t$  put

$$F(s, t) = f \circ \exp(s(X + tV)) \circ \exp(-sX).$$

It is clear that one gets a vector field  $W_s$  defined, for  $s > 0$ , by

$$(2) \quad f' \circ sW_s = (\partial/\partial t)F(s, t)|_{t=0}.$$

On the other hand we have

$$(3) \quad (\partial/\partial s)F(s, t) = f' \circ T(\exp(s(X + tV))) \circ tV \circ \exp(-sX).$$

If we differentiate the line above with respect to  $t$  at  $t = 0$ , then, using the chain rule, all terms drop out except the differentiation of  $tV$ , and we get

$$f' \circ \text{Ad}(\exp sX) V.$$

Comparing the derivative of (2) with respect to  $s$  and the derivative of (3) with respect to  $t$  at  $t = 0$ , one has

$$(d/ds)sf' \circ W_s = f' \circ \text{Ad}(\exp sX) V,$$

and integrating from 0 to 1 gives the conclusion.

*Remark.* It is not necessary that  $\exp X$  be globally defined. The formula of Proposition (1) holds at any point  $p \in \mathcal{M}$  such that the flow of  $X$  with initial condition at  $p$  is defined in a neighborhood of the time interval  $[-1, 0]$ ; this is to say that  $\exp(-X)p$  is defined.

The commutator of vector fields defined on test functions by

$$f' \circ [V, W] = (f' \circ V)' \circ W - (f' \circ W)' \circ V.$$

A straightforward calculation gives

$$(4) \quad (d/ds) \text{Ad}(\exp sX) V = \text{Ad}(\exp sX) [X, V] = [X, \text{Ad}(\exp sX) V].$$

Thus  $\text{ad } X$  defined on  $\text{VEC}(\mathcal{M})$  by  $\text{ad } X V = [X, V]$  is the infinitesimal generator of the one-parameter group of operators  $s \mapsto \text{Ad}(\exp sX)$ . We may therefore write

$$\text{Ad}(\exp sX) \equiv \exp(s \text{ad } X)$$

with the understanding that the right-hand side is to be interpreted in the general case as the left-hand side. On an analytic manifold one may expand

$$(5) \quad \exp(s \text{ad } X) V = \sum_{k=0}^{\infty} (1/k!) s^k (\text{ad } X)^k V.$$

This formula remains valid if there exists a finite-dimensional Lie algebra  $\mathfrak{g} \subset \text{VEC}(\mathcal{M})$  such that  $X, V \in \mathfrak{g}$ . Let us define

$$\psi(u) = \int_0^1 e^{su} ds = u^{-1}(e^u - 1).$$

The formula of (1) may be rewritten to give

(6) **Theorem.** *If  $X, V \in \text{VEC}(\mathcal{M})$  then at any point  $p \in \mathcal{M}$  for which  $\exp(-X)p$  is defined we have (composition with test functions being understood)*

$$(7) \quad (d/dt) \exp(X + tV) \circ \exp(-X)|_{t=0} = \psi(\text{ad } X) V.$$

This formula with  $\psi$  expanded as a power series gives the usual expression, [1, Chapter II, Theorem 1.7]. (The difference in sign results from the fact that we have differentiation acting on the right instead of the left.) In the context of Lie groups, there is a somewhat longer proof in the same spirit as ours in [2, Theorem 2.14.2]. Note that the power series expansion is valid not only on analytic manifolds but also if  $\mathcal{M}$  is merely a  $C^{(2)}$ -manifold provided that  $X$  and  $V$  are  $C^{(2)}$ -vector fields generating a finite-dimensional Lie algebra.

In the application to Lie groups  $\mathbf{G}$  one considers the left multiplication operator  $L_S$  and the right multiplication operator  $R_S$  corresponding to an element  $S \in \mathbf{G}$ . The Lie algebra  $\mathfrak{g}$  is identified as  $T_I(\mathbf{G})$ , and one associates to  $X \in \mathfrak{g}$  the infinitesimal left translation (right-invariant vector field)  $\Lambda_X$  by defining  $\Lambda_X(S) = T(R_S)X$ . The bracket in the Lie algebra is given by  $[X, Y] = [\Lambda_X, \Lambda_Y](I)$ . Since left- and right-multiplication commute, we have  $(\text{Ad } L_S)\Lambda_X = \Lambda_Y$  where  $Y = (\text{Ad } S)X$  in the usual sense of the adjoint action of  $\mathbf{G}$  on  $\mathfrak{g}$ . Thus (7) extends the standard formula for Lie groups to general vector fields on a manifold.

#### REFERENCES

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