ON THE PRIMARINESS OF THE BANACH SPACE \( l_\infty /C_0 \)

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Abstract. In a response to a question asked by Leonard and Whitfield (1983) we show that, under the Continuum Hypothesis, the Banach space \( l_\infty /C_0 \) is primary.

1. Introduction

A Banach space \( E \) is said to be primary if for every direct sum decomposition \( E = X \oplus Y \) of \( E \), at least one of the summands is isomorphic to \( E \). The question of whether or not the Banach space \( C = l_\infty /C_0 \) is primary was raised by Leonard and Whitfield [6], and was motivated by the following two natural decompositions of \( C \): \( C \cong C \oplus C \) and \( C \cong l_\infty \oplus C \). J. Lindenstrauss has shown that \( l_\infty \) is prime [7]. The primary question for Banach spaces has been studied extensively. [2] is a good reference for this subject. The space \( C \) has been used before by Y. Benyamini to construct an example of an \( M \)-space which is not isomorphic to a \( C(K) \) space [1].

In this paper we show that, under the continuum hypothesis (henceforth abbreviated CH), the space \( C \) is primary. We accomplish this in two steps, as in Pelczynski’s classical proof that the spaces \( l_p \) are prime (see [8]): In §2, as a corollary to our result on operators on \( C \), we obtain the following.

(i) If \( C = X \oplus Y \), then \( X \) or \( Y \) contains a complemented copy of \( C \). In §3, we prove that

(ii) \( l_\infty (C) \), the \( l_\infty \)-sum of countably many copies of \( C \), is isomorphic to a complemented subspace of \( C \); in consequence, \( l_\infty (C) \cong C \).

From (i) and (ii), using Pelczynski’s decomposition method, it follows easily that \( C \) is primary.

As is well known (see e.g. [6]), \( C \) can be identified with \( C(\omega^*) \), the space of continuous scalar-valued functions on \( \omega^* = \beta\omega \sim \omega \), the growth of the Stone–Čech compactification of the discrete space \( \omega = \{1, 2, \ldots\} \). In §§2 and 3 we work exclusively with \( C \) thus represented. We do not need CH in (i); however,
our proof of (ii) relies heavily on the fact [10] that open \( F_\alpha \) sets in \( \omega^* \) are retracts of \( \omega^* \) which was established assuming CH, and is now known to be false without CH [4; pp. 205–206]. Of course, it might still be true that the primariness of \( C \) can be established within ZFC though our personal feeling is that this is not the case.

In §4, which concludes the paper, we collect some observations on the space \( l_\infty(C) \). In particular, we show that an isomorphic embedding of this space into \( C \) cannot be obtained as a map induced by an operator from \( l_\infty \) into itself.

Our Banach space terminology and notation are standard, as in [8]; the symbols \( \cong \) and \( \equiv \) are used to denote isomorphisms and isometric isomorphisms between Banach spaces. For the information on the Stone–Cech compactifications the reader is referred to [14] and [9]; basic facts about \( \beta \omega \) and \( \omega^* \) are gathered together on p. 532 of [6].

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2. Operators on \( C = l_\infty/C_0 = C(\omega^*) \)

We start by introducing some notation. If \( A \subset \omega^* \), then \( 1_A \) denotes the characteristic function of \( A \) relative to \( \omega^* \), \( \mathcal{A}(A) \) stands for the algebra of clopen subsets in the subspace \( A \), and \( \mathcal{A}_0(A) = \mathcal{A}(A) \sim \{ \emptyset \} \); we write simply \( \mathcal{A} \) and \( \mathcal{A}_0 \) when \( A = \omega^* \). Recall that if \( A \in \mathcal{A}_0 \), then \( A \) is homeomorphic to \( \omega^* \) and hence \( C(A) \equiv C \); in what follows we often identify \( C(A) \) with the subspace \( \{ x: 1_A x = x \} \) of \( C \). We recall also that the algebra \( \mathcal{A} \) has the following property (called Cantor separability in [14]): For every decreasing sequence \( (A_n) \) in \( \mathcal{A}_0 \), there is an \( A \in \mathcal{A}_0 \) which is contained in all \( A_n \). Finally, if \( T: C \rightarrow C \) is a (bounded linear) operator and \( A \in \mathcal{A} \), then the operator \( T_A: C(A) \rightarrow C(A) \) is defined by \( T_A(x) = T(x)1_A \).

Our main result in this section is the following.

2.1. Theorem. If \( T: C \rightarrow C \) is an operator, then for every \( A \in \mathcal{A}_0 \) there exists a \( B \in \mathcal{A}_0(A) \) and a scalar \( \gamma \) such that

\[
T_B(x) = \gamma x \quad \text{for all } x \in C(B).
\]

This will follow directly from Lemma 2.3 below applied to the representing measure \( \mu \) of \( T: \mu(E) = T(1_E) \) for \( E \in \mathcal{A} \).

We first prove the following (comp. [11; Lemma 1]):

2.2. Lemma. Let \( \lambda: \mathcal{A}_0 \rightarrow R \) be a nondecreasing set function. Then for every \( A \in \mathcal{A}_0 \) there exist \( B \in \mathcal{A}_0(A) \) and \( \beta \in R \) such that

\[
\lambda(E) = \beta \quad \text{for all } E \in \mathcal{A}_0(B).
\]

Proof. Define \( \tilde{\lambda}: \mathcal{A}_0 \rightarrow R \cup \{-\infty\} \) by

\[
\tilde{\lambda}(E) = \inf\{\lambda(F): F \in \mathcal{A}_0(E)\}.
\]
Now, given \( A \in \mathcal{A}_0 \), chose by induction a sequence \( A = A_1 \supset A_2 \supset A_3 \supset \cdots \) in \( \mathcal{A}_0 \) so that, for each \( n \),
\[
\lambda(A_n) \leq \tilde{\lambda}(A_{n-1}) + \frac{1}{n} \quad \text{if} \quad \tilde{\lambda}(A_{n-1}) \neq -\infty ,
\]
or
\[
\lambda(A_n) \leq -n \quad \text{if} \quad \tilde{\lambda}(A_{n-1}) = -\infty .
\]
Since the sequence \( \tilde{\lambda}(A_n) \) is nondecreasing, the limit
\[
\beta = \lim \tilde{\lambda}(A_n) = \lim \lambda(A_n) \text{ exists in } \mathbb{R} \cup \{-\infty\} .
\]
By the Cantor separability of \( \mathcal{A} \), there is a \( B \in \mathcal{A}_0 \) such that \( B \subset A_n \) for all \( n \). Now, if \( E \in \mathcal{A}_0(B) \), then
\[
\tilde{\lambda}(A_n) \leq \lambda(E) \leq \lambda(A_n) \quad \text{for all } n ;
\]
hence \( \lambda(E) = \beta \) (and \( \beta \in \mathbb{R} \)).

2.3. Lemma. Let \( \mu: \mathcal{A} \to C = C(\omega^*) \) be a bounded finitely additive vector measure. Then for every \( A \in \mathcal{A}_0 \) there exists a \( B \in \mathcal{A}_0(A) \) and a scalar \( \gamma \) such that
\[
1_B \mu(E) = \gamma 1_E \quad \text{for all } E \in \mathcal{A}(B) .
\]
Proof. Clearly, it suffices to prove this for the real space \( C \). Define a nondecreasing set function \( \lambda, \mu: \mathcal{A} \to \mathbb{R} \) by
\[
\lambda_\mu(E) = \sup F \in \mathcal{A}(E) \max F ,
\]
and define \( \lambda_{-\mu} \) likewise.

Let \( A \in \mathcal{A}_0 \). Applying Lemma 2.2 first to \( \lambda = \lambda_\mu \), and next to \( \lambda = \lambda_{-\mu} \), we find \( B \in \mathcal{A}_0(A) \) and scalars \( \alpha, \beta \) such that
\[
\lambda_\mu(E) = \beta
\]
and
\[
(*) \quad -\lambda_{-\mu}(E) = \inf E \in \mathcal{A}(E) \min \mu(F) = \alpha \quad \text{for all } E \in \mathcal{A}_0(B) .
\]
Clearly, \( \alpha \leq 0 \leq \beta \). If \( \alpha = \beta = 0 \), then \( 1_B \mu(E) = 0 \cdot 1_E \) for all \( E \in \mathcal{A}(B) \). Now let us consider the case when \( \alpha < 0 \) or \( \beta > 0 \); by switching from \( \mu \) to \( -\mu \) if necessary, we may assume \( \beta > 0 \).

Claim \( \alpha = 0 \).

Suppose \( \alpha < 0 \), and let \( 0 < \epsilon < \frac{1}{2} \min(-\alpha, \beta) \). Since \( \lambda_\mu(B) = \beta \), there exists \( E \in \mathcal{A}(B) \) such that \( \max_B \mu(E) > \beta - \epsilon \), and hence there exists an \( H \in \mathcal{A}(B) \) with
\[
\mu(E) > \beta - \epsilon \quad \text{on } H .
\]
Observe that \( H \subset E \): In fact, if \( H \sim E \neq \emptyset \), then \( \lambda_\mu(H \sim E) = \beta \) and we could find \( F \in \mathcal{A}(H \sim E) \) so that \( \max_{H \sim E} \mu(F) > \beta - \epsilon \). But in that case \( \max_H \mu(E \cup F) > 2(\beta - \epsilon) > \beta \), which is impossible. Next, using \((*)\),
choose $F \in \mathcal{A}(H)$ with $\min_H \mu(F) < \alpha/2$. Then $\max_H \mu(E \sim F) = \max_H [\mu(E) - \mu(F)] > \beta + \alpha/2 - \alpha/2$ and so $\lambda_\mu(E) > \beta$, which is impossible.

From the above it now follows that the vector measure $\mu_B : \mathcal{A}(B) \rightarrow C$ defined by $\mu_B(E) = 1_B \mu(E)$ is nonnegative and hence monotone:

(1) If $E, F \in \mathcal{A}(B)$ and $F \subset E$, then $0 \leq \mu_B(F) \leq \mu_B(E)$.

From this and the definition of $\lambda_\mu$ it follows immediately that

(2) $\max 1_B \mu(E) = \max 1_E \mu(E) = \beta$ for all $E \in \mathcal{A}_0(B)$.

Now we show that

(3) $\mu_B(E) \leq \beta 1_E$ for all $E \in \mathcal{A}(B)$.

Suppose it is not so for some $E \in \mathcal{A}_0(B)$; thus there is an $H \in \mathcal{A}_0(B \sim E)$ and an $\varepsilon > 0$ such that $\mu_B(E) > \varepsilon 1_H$. Then, as $\max 1_H \mu(H) = \beta$, we have

\[ \max 1_H \mu(E \cup H) = \max [1_H \mu(E) + 1_H \mu(H)] \geq \max [\varepsilon + 1_H \mu(H)] = \varepsilon + \beta; \]

a contradiction.

From (1), (2), and (3) it now follows that $\mu_B(E) = \beta 1_E$ for all $E \in \mathcal{A}(B)$, which concludes the proof.

2.4. Corollary. Let $I$ be the identity operator on $C$, and let $(T_n)$ be a (finite or infinite) sequence of operators on $C$ such that

\[ I = T_1 + T_2 + \cdots \text{ pointwise on } C. \]

Then for every $A \in \mathcal{A}_0$ there exists a $B \in \mathcal{A}_0(A)$ and an $m$ such that $T_m$ maps $C(B)$ isomorphically onto a complemented subspace of $C$.

Proof. By Theorem 2.1 we can find a decreasing sequence $(B_n)$ in $\mathcal{A}_0$, with $B_1 \subset A$, and a sequence of scalars $(\gamma_n)$ so that for each $n$,

\[ T_n(x)1_{B_n} = \gamma_n x \text{ for all } x \in C(B_n). \]

By the Cantor separability of $\mathcal{A}$ there exists $B \in \mathcal{A}_0$ which is contained in all $B_n$'s. Then

\[ T_n(x)1_B = \gamma_n x \text{ for all } x \in C(B) \text{ and all } n. \]

Moreover, by assumption we have $x = I(x)1_B = \gamma_1 x + \gamma_2 x + \cdots$ for all $x \in C(B)$; therefore, $\gamma_m \neq 0$ for some $m$. Since

\[ ||\gamma_m|| ||x|| \leq ||T_m(x)|| \leq ||T_m|| ||x|| \text{ for } x \in C(B), \ T_m|_{C(B)} \]

is an isomorphism. Finally, the map $P : C \rightarrow C$ defined by $P(x) = T_m(x1_B)$ is a projection onto $T_m|_{C(B)}$.

2.5. Corollary. For every (finite or infinite) Schauder decomposition $C = X_1 + X_2 + \cdots$ of $C$ at least one of the summands $X_n$ contains a subspace isomorphic to $C$ and complemented in $C$. 

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2.6. **Remark.** It is also known ([6; 5.1], [13; Prop. 2]) that every infinite-dimensional complemented subspace of \( C \) contains an isomorphic copy of \( l_\infty \).

### 3. \( C \) is primary

Throughout this section we assume the continuum hypothesis (CH). The following result is due to Negrepontis [10; Corollary 3.2], and it cannot be proved without CH [4].

3.1. **Theorem (CH).** If \( A \) is an open \( F_\sigma \) subset of \( \omega^* \), then \( \overline{A} \) is a retract of \( \omega^* \).

(Equivalently, there is a Boolean \( F_\sigma \) subset of \( \omega^* \) such that \( \sigma(B) \cap A = B \) for all \( B \in \mathcal{A}(A) \).)

3.2. **Proposition (CH).** \( l_\infty(C) \) is isometric to a complemented subspace of \( C \). As a consequence \( \mathcal{L}(C) \cong C \).

**Proof.** Let \( (A_n) \) be an infinite sequence of disjoint nonempty clopen subsets of \( \omega^* \) and let \( A \) be its union. Then \( l_\infty(C) \) is isomorphically isometric to the Banach space \( C_b(A) \) of bounded continuous functions on \( A \). Moreover, as is easily seen \( \overline{A} = \beta A \), and therefore \( C_b(\overline{A}) \cong C(A) \). Hence \( l_\infty(C) \cong C(A) \). Now, by Theorem 3.1, there is a retraction \( r \) of \( \omega^* \) onto \( \overline{A} \). Then the corresponding composition operator \( R: C(\overline{A}) \to C; \ x \mapsto x \circ r \) is an isometry, and the operator \( P \) defined by \( P(x) = R(x|_{\overline{A}}) \) is a projection from \( C \) onto \( R[C(\overline{A})] \).

The second assertion follows easily from the first by using Pelczynski's decomposition method: Let \( C \cong l_\infty(C) \oplus Z \); then

\[
l_\infty(C) \cong l_\infty(C) \oplus C \cong l_\infty(C) \oplus l_\infty(C) \oplus Z \cong l_\infty(C) \oplus Z \cong C.
\]

3.3. **Theorem (CH).** Let \( C = X_1 \oplus X_2 \oplus \cdots \) be a (finite or infinite) Schauder decomposition of \( C \). Then there is an \( m \) such that \( X_m \approx C \). In particular, \( C \) is primary.

**Proof.** By Corollary 2.5 there is an \( m \) such that \( X_m \) contains a subspace \( V \) which is isomorphic to \( C \) and complemented in \( C \). Assuming, as we may, that \( m = 1 \), and denoting \( X = X_1 \), \( Y = X_2 \oplus X_3 \oplus \cdots \), we thus have

\[
C = X \oplus Y, \quad X = U \oplus V, \quad \text{and} \quad V \approx C.
\]

(for some subspace \( U \) of \( X \)). Moreover, by Proposition 3.2, \( C \cong l_\infty(C) \). Now, applying Pelczynski's decomposition method,

\[
X \approx U \oplus l_\infty(C) \approx U \oplus C \oplus l_\infty(C) \approx X \oplus l_\infty(X \oplus Y)
\]

\[
\approx X \oplus l_\infty(X) \oplus l_\infty(Y) \approx l_\infty(X) \oplus l_\infty(Y) \approx l_\infty(X \oplus Y) \approx l_\infty(C) \cong C.
\]

3.4. **Remarks.** (1) In view of the natural decompositions \( C \cong C \oplus C \) and \( C \cong l_\infty(C) \), one might conjecture that an infinite-dimensional complemented
subspace of $C$ must be isomorphic either to $l_\infty$ or to $C$. It is however easy to see that it is not so if we assume CH: Let $L = L_\infty([0, 1]^c)$, where $C = 2^{c_0} = \chi_1$ and $[0, 1]^c$ is considered with its product Lebesgue measure. Then the measure algebra of $[0, 1]^c$ has cardinality $c = \chi_1$; moreover, by Parovicenko's theorem [14; p. 81], every Boolean algebra of cardinality $\leq \chi_1$ can be embedded via a Boolean isomorphism into $\mathscr{P}(\omega^*)$. From this it follows that $L$ is isometric to a closed subspace $X$ of $C$. Since $L$ is an injective Banach space, $X$ is complemented in $C$. Moreover, $L \neq l_\infty$ because no sequence of continuous linear functionals separates points in $L$, and $L \approx C$ because $C$ is not injective (see [12]). (By Theorem 3.3, every complement of $X$ in $C$ is isomorphic to $C$.)

(2) The proof of the decomposition $C \approx l_\infty \oplus C$ given in [6; 5.4] uses implicitly CH, which would be easily avoided: $C \approx (l_\infty \oplus l_\infty)/\{0\} \oplus C_0 \approx l_\infty \oplus C$, where the first isomorphism is justified by the separable homogeneity of $l_\infty$ [8; 2.f.12(i) and p. 112].

4. SOME OBSERVATIONS ON THE SPACE $l_\infty(C)$

In the proof of Proposition 3.2 we represented $l_\infty(C)$ isometrically in the form $C_b(A_1 \cup A_2 \cup \cdots)$, where $(A_k)$ was any disjoint sequence of nonempty clopen sets in $\omega^*$. The corresponding “quotient” representations of $l_\infty(C)$ are of course of the form

$$l_\infty/\left(\sum_k C_0[N_k]\right)_{l_\infty},$$

where $(N_k)$ is a partition of $\omega$ into a sequence of infinite subsets. Here for $M \subset \omega$ we let $l_\infty[M] = \{x = (x_n) \in l_\infty: x_n = 0$ for $n \notin M\}$, $C_0[M] = C_0 \cap l_\infty[M]$, and we identify $(\sum l_\infty[N_k])_{l_\infty}$ in a natural manner with $l_\infty$, and $(\sum C_0[N_k])_{l_\infty}$ with subspace of $l_\infty$. Clearly, the latter space is isometrically isomorphic to $l_\infty(C_0)$. In Corollary 4.2 below we show that whenever $X$ is a subspace of $l_\infty$ isomorphic to $l_\infty(C_0)$, then $l_\infty/X \approx l_\infty(C)$. We digress momentarily to recall some results concerning $l_\infty$.

(A) If $X$ and $Y$ are isomorphic closed subspaces of $l_\infty$ such that both $l_\infty/X$ and $l_\infty/Y$ are nonreflexive, then every isomorphism between $X$ and $Y$ extends to an automorphism of $l_\infty$; in particular, $l_\infty/X \approx l_\infty/Y$ [8; 2.f.12(i)].

(B) If $X$ is a closed subspace of $l_\infty$, then $l_\infty/X$ is nonreflexive if and only if $l_\infty/X$ contains an (isomorphic) copy of $l_\infty$ [8; the proof of 2.f.12(i)].

(C) If a Banach space $E$ has a subspace isomorphic to $l_\infty$, and $X$ is a closed subspace of $E$, then $X$ or $E/X$ must contain a copy of $l_\infty$ [5].

4.1. Proposition. Let $X$ be a closed subspace of $l_\infty$. Then $l_\infty/X$ is nonreflexive in each of the following cases:

(a) $X \neq l_\infty$ (i.e. $X$ contains no copy of $l_\infty$),

(b) $X = U \oplus V$, where $U \approx l_\infty$, $\dim V = \infty$, and $V \neq l_\infty$,

(c) $X = U \oplus V$, where $V$ is nonreflexive and $V \neq l_\infty$. 

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Proof. (a) See [8, 2.1.13].

Let $Q: l_\infty \to l_\infty /U$ be the quotient map, and note that

$$l_\infty /X \approx (l_\infty /U)/Q(V).$$

(b) Since $\dim(l_\infty /U) = \infty$ and $U \approx l_\infty$, $l_\infty /U \approx l_\infty$ ([8, 2.1.7]); moreover, $Q(V) \approx V \not\approx l_\infty$, hence $l_\infty /X$ is nonreflexive by (a) (or (C)).

(c) Since $l_\infty /U \supset Q(V) \approx V$ and $V$ is nonreflexive, neither is $l_\infty /U$. Hence, by (B), $l_\infty /U \not\approx l_\infty$. But $Q(V) \not\approx l_\infty$, hence $l_\infty /X$ must contain a copy of $l_\infty$, by (C).

4.2. Corollary. Let $X$ be a subspace of $l_\infty$ isomorphic to $l_\infty(C_0)$. Then

$$l_\infty /X \not\approx l_\infty(C).$$

Proof. If $Y = (\sum C_0[N_k])_{l_\infty} \subset l_\infty$, as at the beginning of this section, then $Y \approx l_\infty(C_0)$ and $l_\infty /Y \approx l_\infty(C)$. Since $l_\infty(C_0) \cong l_\infty(C) \oplus C_0$, it is enough to apply (A) and Proposition 4.1(c).

From Proposition 3.2 we know that, under CH, $l_\infty(C)$ is isomorphic to a subspace of $C$. We are going to show now that such an isomorphic embedding cannot be induced by an operator of $l_\infty$ into itself.

4.3. Proposition. Let $X$ and $Y$ be subspaces of $l_\infty$, with $X \cong l_\infty(C_0)$ and $Y \cong C_0$. There exists no operator $T: l_\infty \to l_\infty$ mapping $X$ into $Y$ and such that the induced operator $\tilde{T}: l_\infty /X \to l_\infty /Y$ is an isomorphic embedding.

Proof. Recall that if $Q: l_\infty \to l_\infty /X$ and $q: l_\infty \to l_\infty /Y$ are the quotient maps then $T$ is the unique operator for which $TQ = qT$.

In view of Corollary 4.2, Proposition 4.1(a), and (A) (see [8, p. 211]), we may assume that $X = (\sum C_0[N_k])_{l_\infty}$ (as at the beginning of this section) and $Y = C_0$. In what follows we denote $X$ thus represented simply by $l_\infty(C_0)$.

Now, let $T: l_\infty \to l_\infty$ be an operator such that $T[l_\infty(C_0)] \subset C_0$, and let $f_j \in (l_\infty)^*$ be the coordinate maps of $T$: $T(x) = (f_j(x))$ for every $x \in l_\infty$. For each $j$ let $\mu_j$ be the bounded finitely additive measure on the power set $P(\omega)$ of $\omega$ representing $f_j$. Note that $\sup_j |\mu_j(A)| = \|T(1_A)\|$ for all $A \subset \omega$.

(*) Claim. For every $\varepsilon > 0$, there exists a $k$ such that whenever $A$ is a finite subset of $N_k \cup N_{k+1} \cup \cdots$, then $\|T(1_A)\| < \varepsilon$.

Suppose it is not so. Then we can find an $\varepsilon > 0$, a strictly increasing sequence $(k_n)$ of indices, and a sequence $(A_n)$ of finite sets such that, for each $n$, $A_n \subset U[N_k: k_n \leq k < k_{n+1}]$ and $\|T(1_{A_n})\| \geq \varepsilon$.

Since $l_\infty \cong (\sum l_\infty[A_n])_{l_\infty} \subset l_\infty(C_0)$, a result of H. P. Rosenthal [9] applied to the operator $T|(\sum l_\infty[A_n])_{l_\infty}: (\sum l_\infty[A_n])_{l_\infty} \to C_0$ shows that this operator must be an isomorphism on a subspace isomorphic to $l_\infty$; a contradiction.

Now, for each $k$, pick an infinite subset $M_k$ of $N_k$ so that $\mu_j(M_k)$ is $\sigma$-additive for all $j$ (see [1; p.38]). Then from (*) it follows that (**) for every
\( \varepsilon > 0 \), there exists a \( k \) such that if \( p \geq K \) and \( A \) is any subset of \( M_k \cup \cdots \cup M_p \), then \( \| T(1_A) \| < \varepsilon \).

Suppose \( \tilde{T} \) is an isomorphic embedding. Then
\[
\eta = \inf\{\| \tilde{T}(Qx) \| : \| Qx \| = 1 \} > 0;
\]
in particular, since \( \tilde{T}Q = qT \), we must have
\[
\| T(1_A) \| \geq \| q(T(1_A)) \| = \| \tilde{T}(Q1_A) \| \geq \eta
\]
for every set \( A \subset \omega \) that has an infinite intersection with at least one of the sets \( N_k \). It follows that \( \| T(1_{M_k}) \| \geq \eta \) for every \( k \), contradicting (**)..

4.4. Proposition (CH). \( l^\infty(C)/C_0(C) \cong C \).

Proof. As in the proof of Proposition 3.2, let \( A \) be the union of a sequence \( (A_n) \) of pairwise disjoint nonempty clopen subsets of \( \omega^* \). Then, identifying \( l^\infty(C) \) with \((\sum C(A_n))_{l^\infty}\), there is a natural isometric isomorphism between \( l^\infty(C) \) and \( C(\overline{A}) \) in which the function \( f \in C(\overline{A}) \) corresponding to an element \( (f_n) \) of \( l^\infty(C) \) is such that \( f|A_n = f_n \) for all \( n \). Under this isomorphism the subspace \( C_0(C) \) of \( l^\infty(C) \) is mapped onto the subspace in \( C(\overline{A}) \) consisting of functions vanishing on the boundary \( \partial A = \overline{A} \sim A \) of \( A \). It follows that \( l^\infty(C)/C_0(C) \cong C(\partial A) \). Finally, by a result of Gillman [14; p. 161], assuming CH we have that \( \partial A \) is homeomorphic with \( \omega^* \) and therefore \( C(\partial A) \cong C \).

We conclude with a few remarks.

We used CH only to show that \( l^\infty/c_0 \approx (\sum l^\infty/c_0)_{l^\infty} \).

It may be possible to show that \( l^\infty/c_0 \) is primary without this result and therefore without CH. However, it is likely that \( l^\infty/c_0 \) being primary implies this decomposition. For example, in [2] it is shown that \( X = (\Sigma \oplus E_n)_p \), \( \dim E_n < \infty \), is primary implies \( X \approx (\Sigma \oplus X)_p \). A variation of this technique may show that the \( l^\infty \)-decomposition of \( l^\infty/c_0 \) is necessary (as well as sufficient) for \( l^\infty/c_0 \) to be primary. A famous open problem in Banach space theory is the Schroder-Bernstein problem: If two Banach spaces \( X \) and \( Y \) embed complementably into one another, must \( X \) and \( Y \) be isomorphic? We have shown that whenever \( l^\infty/c_0 \approx X \oplus Y \) then \( l^\infty/c_0 \) embeds complementably into either \( X \) or \( Y \) (without using CH). So if \( l^\infty/c_0 \) fails to be primary without CH, then it is also a counterexample to the Schroder-Bernstein property. This leaves open the possibility that the Schroder-Bernstein problem has a negative answer in ZF but a positive answer with CH.

References

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