RANDOM PERTURBATIONS OF SINGULAR SPECTRA

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Abstract. The singular parts of the self-adjoint operators $T$ and $H = T + V$ are mutually singular for "almost every" bounded perturbation $V$.

Let $T$ and $H = T + V$ be self-adjoint operators on a separable Hilbert space $\mathcal{H}$. In 1965, Donaghue [1] proved the following theorem.

Theorem 1. If $V = c \langle \cdot, \varphi \rangle \varphi$ is of rank one, and $\varphi$ is $T$-cyclic, then the singular parts of $T$ and $H$ are supported on disjoint sets.

We shall generalize this result by proving that the singular parts of $T$ and $H = T + V$ are mutually singular for "almost every" bounded perturbation $V$. This theorem, which follows easily by the methods of Simon-Wolff [5] and the author [2], illustrates an essential instability of the singular spectrum.

Let $T$ be self-adjoint, $\varphi_n$ an arbitrary complete orthonormal set, $c_n > 0$ an arbitrary bounded sequence of positive numbers, and $X_n(\omega)$ a sequence of independent random variables, uniformly distributed on $[-1, 1]$. Define

$$H(\omega) = T + V(\omega),$$

where

$$V(\omega) = \sum_{n=1}^{\infty} c_n X_n(\omega) \langle \cdot, \varphi_n \rangle \varphi_n.$$

Theorem 2. The singular parts $T$ and $H(\omega)$ are supported on disjoint sets, almost surely.

Proof. We sketch the proof, which follows [2]. Let $N$ be a set of Lebesgue measure zero which supports the singular part of $T$. Define the multiplication operator

$$Hu(\omega) = H(\omega)u(\omega)$$

on $L_2(\Omega, P; \mathcal{H})$. □

Lemma. $H$ is spectrally absolutely continuous.

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Assuming this for the moment, and taking $u(\omega) = u$ to be a constant function, we have

$$0 = |E[N]u|^2 = \int_\Omega |E_\omega[N]u|^2 P(d\omega),$$

and hence

$$E_\omega[N]u = 0, \quad \text{a.s.}$$

Letting $u$ range over a countable dense set gives

$$E_\omega[N] = 0, \quad \text{a.s.},$$

which is the result. (This argument is essentially due to Kotani.)

The following proof of the lemma is slightly more elementary than that of [2], so we have included it here.

Proof. Let $Q$ be the coordinate operator of multiplication by $x$ on $L^2(\mathbb{R})$ and

$$P = -i \frac{d}{dx}$$

its conjugate. If $f(t)$ is a bounded smooth function with $f'(t) > 0$ (for example, $\arctan(t)$), then because, under the Fourier transformation

$$Q = i \frac{d}{dp},$$

we have

$$i[f(P), Q] = f'(P) \geq 0.$$  

If $\chi(x)$ is the characteristic function of $[-1, 1]$, then

$$i[\chi(Q)f(P), \chi(Q)Q], \chi(Q)Q = \chi(Q)\overline{f'(P)}\chi(Q) > 0.$$  

Thus, the operator $X = \chi(Q)Q$ of multiplication by $x$ on $L^2[-1, 1]$ has a positive commutator with the bounded operator,

$$B = \chi(Q)f(P)\chi(Q).$$  

Now, representing $\Omega$ as the infinite product of $[-1, 1]$'s, we have that

$$L^2(\Omega; \mathcal{H}) = \left( \bigotimes_{n=1}^\infty L^2[-1, 1] \right) \otimes \mathcal{H}.$$  

Each $X_n(\omega)$ becomes multiplication by $\chi(Q)Q$ on one factor of the tensor product. Let $B_n$ be as above with

$$i[B_n, X_n] > 0,$$

and define

$$A = \sum_{n=1}^\infty \frac{1}{2^n} B_n.$$  

Then

$$i[A, H] = i \sum_{n=1}^\infty c_n[B_n, X_n] \otimes \langle \cdot, \varphi_n \rangle \varphi_n \geq 0.$$
Hence, $H$ has a positive commutator with a bounded operator of dense range and is therefore absolutely continuous by the Kato–Putnam theorem [4, p. 157]. □

**Remarks.** (1) One could replace the uniform distribution by other bounded, absolutely continuous distributions [2, p. 67]. In fact, as long as $X_n$ are independent, one could presumably relax the requirement that they be identically distributed.

(2) One can take $c_j$ tending very rapidly to zero, or not tending to zero at all, so that $V(\omega)$ is a very general sort of diagonal operator. This is rather satisfactory if one recalls that, by the Weyl-von Neumann Theorem [3, p. 523], every bounded self-adjoint operator differs from such an operator by an operator of arbitrarily small Hilbert-Schmidt norm.

**References**


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