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STRONG SOLUTIONS OF EVOLUTION EQUATIONS GOVERNED BY m-ACCRETIVE OPERATORS AND THE RADON-NIKODYM PROPERTY

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ABSTRACT. We construct, in every Banach space which fails the Radon-Nikodym property, a nonlinear operator A which is m-accretive for some equivalent norm in X, such that the domain of A is not a singleton and such that the only strong solutions of the equation $u' + Au \ni f$ are the constant ones.

1. INTRODUCTION

Let X be a Banach space. An operator A in X is a map from X into the power set 2^X of X. Let us denote

$$D(A) = \{x \in X : Ax \neq \varnothing\}$$

the domain of A, and

$$R(A) = \{y \in X : \exists x \in X \text{ such that } y \in Ax\}$$

the range of A. We shall freely identify A with the subset of $X \times X$ of all couples (x, y) such that $x \in D(A)$ and $y \in Ax$. The operator A is called accretive if for every (x, y) (x', y') in A and for every $\lambda \geqq 0$:

$$\|x - x' + \lambda(y - y')\| \geq \|x - y\|.$$

The operator A is said to be m-accretive if, moreover, for every $\lambda > 0$

$$R(I + \lambda A) = X,$$

where I stands for the identity operator in X. It is well known that A is m-accretive if and only if $R(I + \lambda A) = X$ for some $\lambda > 0$.

The operator A is said to be trivial if there exists $a \in X$ such that $A = \{a\} \times X$.

Now let us consider the equation:

$$u' + Au \ni f, \tag{1}$$

where A is m-accretive and $f \in L^1(0, T; X)$. We shall say that u is a

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strong solution of (1) if u is absolutely continuous and differentiable almost everywhere on $[0, T]$ and if

$$u' + Au(t) \ni f(t) \quad \text{for almost every } t \in [0, T].$$

Observe that if $(x, y) \in A$ and $f(t) = y$ for almost every $t \in [0, T]$ then the function u which is constant equal to x is a strong solution of (1) on $[0, T]$; in this case u is called a constant solution of (1) on $[0, T]$. We shall see below that in some cases the only strong solutions of (1) on $[0, T]$ are the constant ones.

A Banach space X has the Radon-Nikodym property if every bounded linear operator T from $L^1([0, 1])$ into X is representable, which means that there exists $g \in L^\infty([0, 1]; X)$ such that for $f \in L^1([0, 1])$, $Tf = \int_0^1 f(t)g(t)\,dt$. The classical Radon-Nikodym theorem expresses the fact that $\mathbf{R}$ has the Radon-Nikodym property. This property has been extensively studied in recent years (see e.g. [6] or [10]). Let us mention that every reflexive Banach space has the Radon-Nikodym property. On the other hand, $L^1(\mathbf{R}^N)$ and $BUC(\mathbf{R}^N)$ (the space of bounded uniformly continuous functions on $\mathbf{R}^N$) fail the Radon-Nikodym property.

Let us assume that A is an m-accretive operator in a Banach space X. Then it is known that, if u_0 belongs to $D(A)$, f is in $BV(0, T; X)$ and X has the Radon-Nikodym property, the equation (1) has a unique strong solution on $[0, T]$ satisfying $u(0) = u_0$ (see [2] or [4]). It is also known that this result is no longer true in some concrete examples where X fails the Radon-Nikodym property. For instance, let us consider the Hamilton–Jacobi equation in $\mathbf{R}^N$:

$$(HJ)\ u_t + H(\nabla u) = f.$$

Under a suitable hypothesis on H we associate to this equation an m-accretive operator A in $BUC(\mathbf{R}^N)$. However, even if the initial data u_0 is smooth ($u_0 \in D(A)$) and $f = 0$, there is no strong solution of (HJ) on $[0, T]$ for every $T > 0$. A similar remark holds for the nonlinear conservation laws in $\mathbf{R}^N$ to which one can associate an m-accretive operator in $L^1(\mathbf{R}^N)$ (see e.g. [2, 4, 8]).

In this note, we prove that in every Banach space which fails the Radon-Nikodym property, there exists a nontrivial operator A in X, which is m-accretive for some equivalent norm in X, such that the only strong solutions of (1) on $[0, T]$ are the constant ones. In particular, if f is not constant almost everywhere, or if f is constant equal to $y \notin R(A)$ then equation (1) has no strong solution. The construction of the operator A relies on the existence, for every Banach space Y which fails the Radon-Nikodym property, of a Lipschitz function ϕ from $\mathbf{R}$ into Y such that the set of all points of differentiability of ϕ is negligible. This last result will be proved in the appendix.

We also give various consequences of the existence of such an operator A, some of them having been already announced in [9].

2. Construction of an m-accretive operator

In this section we prove

Theorem 2.1. *Let X be a Banach space. X has the Radon-Nikodym property if and only if for every equivalent norm $\| \ \|$ on X and for every nontrivial m-accretive operator A in $(X, \| \ \|)$ there exists $f \in L^1(0, T; X)$ (where $T > 0$ is arbitrary) such that* (1) *has a nonconstant strong solution on* $[0, T]$.

We need the following elementary lemma whose proof is left to the reader. This is for instance an immediate consequence of Theorem 8-16 and of corollary of Theorem 8-6 of [11].

Lemma 2.2. *Let I be an interval of $\mathbf{R}$ and f be an absolutely continuous function from I into $\mathbf{R}$. If $A = \{x \in I : f'(x) = 0\}$, then $\mu(f(A)) = 0$.*

Proof of Theorem 2.1.

First, let assume that X has the Radon-Nikodym property. Let $\| \ \|$ be an equivalent norm on X and A be a nontrivial m-accretive operator in $(X, \| \ \|)$. Since A is m-accretive, $D(A)$ is nonempty. Let us pick a in $D(A)$. If $Aa = X$ then, since A is accretive $D(A) = \{a\}$ and so $A = \{a\} \times X$, which contradicts the assumption A nontrivial. Hence, $Aa \neq X$ and we can choose $b \in X \setminus Aa$. Since X has the Radon-Nikodym property, the theory of nonlinear semigroups (see e.g. [2] or [4]) tells us that the initial value problem:

$$\begin{cases} u' + Au \ni b, \\ u(0) = a \end{cases}$$

has a strong solution and this solution is nonconstant since $b \notin Aa$.

Conversely, let X be a Banach space which fails the Radon-Nikodym property. We shall construct an equivalent norm $\| \ \|$ on X and a nontrivial m-accretive operator A in $(X, \| \ \|)$ such that for every $T > 0$ and every $f \in L^1(0, T; X)$, the only strong solutions of $u' + Au \ni f$ are the constant ones.

Step 1. Construction of A. Let $|\cdot|$ be the original norm on X, $x_0 \in X$ and $f \in X^*$ such that $f(x_0) = \|x_0\| = \|f\| = 1$. Since $Y = \operatorname{Ker} f$ is a hyperplane of X and $x_0 \notin Y$, for every $z \in X$, there exists $a \in \mathbf{R}$ and $y \in Y$ such that $z = ax_0 + y$, and we then define $\|z\| = \max(|a|, |y|)$. Clearly $\| \ \|$ is an equivalent norm on X. By Corollary 3.3, there exists a Lipschitz function ϕ from $\mathbf{R}$ into Y such that the set of all points of differentiability of ϕ is negligible. Without loss of generality, we can assume $\phi(0) = 0$ and ϕ is 1-Lipschitz. Finally, define

$$A = \{(tx_0 + \phi(t), y) : t \in \mathbf{R}, \ y \in Y\}.$$

Step 2. A is m-accretive in $(X, \| \ \|)$. Let $(x, y), (x', y') \in A$ and $\lambda > 0$. From the definition of A, there exists $t, t' \in \mathbf{R}$ such that $x = tx_0 + \phi(t)$,

$x' = t'x_0 + \phi(t')$, and $y, y' \in Y$. Thus

$$\begin{aligned}\|x - x'\| &= \|(t - t')x_0 + \phi(t) - \phi(t')\| \\ &= \sup(|t - t'|, |\phi(t) - \phi(t')|) = |t - t'|\end{aligned}$$

and

$$\begin{aligned}\|x - x' + \lambda(y - y')\| &= \|(t - t')x_0 + \phi(t) - \phi(t') + \lambda(y - y')\| \\ &= \sup(|t - t'|, |\phi(t) - \phi(t') + \lambda(y - y')|) \\ &\geqq |t - t'|,\end{aligned}$$

therefore, A is accretive. On the other hand, since, for $\lambda > 0$:

$$\begin{aligned}I + \lambda A &= \{(tx_0 + \phi(t), tx_0 + \phi(t) + \lambda y) : t \in \mathbf{R},\ y \in Y\} \\ &= \{(tx_0 + \phi(t), tx_0 + y) : t \in \mathbf{R},\ y \in Y\}\end{aligned}$$

we have that, if $z = tx_0 + y$ with $t \in \mathbf{R}$ and $y \in Y$, then $z \in (I + \lambda A)(tx_0 + \phi(t))$; thus $R(I + \lambda A) = X$ and A is m-accretive.

Step 3. The only strong solutions of (1) are the constant solutions. Let $u : [0, T] \to X$ be a strong solution of (1) that is not constant. Thus there exists $v \in \mathrm{AC}(0, T; \mathbf{R})$ (the space of absolutely continuous functions from $[0, T]$ into $\mathbf{R}$) and $w \in \mathrm{AC}(0, T; Y)$ such that $u(t) = v(t)x_0 + w(t)$. Since for every $t \in [0, T]$, $u(t) \in \overline{D(A)} = D(A) = \{tx_0 + \phi(t) : t \in \mathbf{R}\}$, we have $w(t) = \phi(v(t))$. Since $w(t) = u(t) - v(t)x_0$ for every $t \in [0, T]$ and since u and v are differentiable almost everywhere, w is differentiable almost everywhere. On the other hand, if $B = \{t \in [0, T]$: either $v'(t)$ does not exist or $v'(t) = 0\}$ then by Lemma 2.2, $\mu(v(B)) = 0$. So, since v is not constant, if $C = \{t \in [0, T] : v'(t)$ exists and $v'(t) \neq 0\}$, then $\mu(C) \neq 0$ and $\mu(v(C)) \neq 0$. Therefore the set

$$A = \{t \in C\colon v(t) \text{ is not a point of differentiability of } \phi\}$$

is not negligible and we claim that w is not differentiable at any point of A which contradicts the fact that w is differentiable almost everywhere. To prove our claim, let $t \in A$. Since for every $\varepsilon > 0$, $\{v(t + h) : |h| < \varepsilon\}$ is a neighborhood of $v(t)$ (use the continuity of v and the fact that $v'(t) \neq 0$), we have that

$$\lim_{h \to 0} \frac{\phi(v(t + h)) - \phi(v(t))}{v(t + h) - v(t)}$$

does not exist. Now we write

$$\frac{w(t + h) - w(t)}{h} = \frac{\phi(v(t + h)) - \phi(v(t))}{v(t + h) - v(t)} \cdot \frac{v(t + h) - v(t)}{h}.$$

This makes sense if $|h|$ is small enough (since then $v(t + h) - v(t) \neq 0$) and observing that $\lim_{h \to 0}(v(t + h) - v(t))/h$ exists and is nonzero, we get that $\lim_{h \to 0}(w(t + h) - w(t))/h$ does not exist, which proves our claim.

Remarks 2.3. 1. The idea of the proof of Theorem 2.1 is actually very simple: we have constructed an m-accretive operator A in $(X, \| \ \|)$ such that the domain of A is the graph of a Lipschitz function which is differentiable only

on a set of Lebesgue measure zero. A solution of (1) is then forced to follow this graph and thus cannot be differentiable almost everywhere if it is not constant.

2. If we replace A by B:

$$B = \{(tx_0 + \phi(t), -x_0 + y) : t \in \mathbf{R},\ y \in Y\}$$

then B is also m-accretive in $(X, \| \|)$ and the only strong solutions of $u' + Bu \ni f$ are the constant solutions. And since $0 \notin R(B)$ the equation $u' + Bu \ni 0$ has no strong solution: this result was obtained in [9]. Let us notice that since the equation $u' + Bu \ni 0$ always has at least one strong solution whenever X has the Radon-Nikodym property and B is m-accretive, we have obtained a new characterization of the Radon-Nikodym property.

3. Note that the Radon-Nikodym property is an isomorphic property i.e. does not depend on the equivalent norm on X. This justifies the fact that we have chosen an equivalent norm on X in the statement of Theorem 2.1. In the linear case, the operators which are m-accretive for some equivalent norm are precisely the generators of uniformly bounded semigroups. The class of nonlinear operators which are m-accretive for some equivalent norm does not seem to be well known.

4. If X is a Banach space, A an m-accretive operator in X, $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; X)$, we say that $u \in \mathscr{C}(0, T; X)$ is a weak solution of

$$\begin{cases} u' + Au \ni f, \\ u(0) = u_0 \end{cases} \tag{2}$$

if there exists $u_{0,n} \in X$, $f_n \in L^1(0, T; X)$, and $u_n \in \mathscr{C}(0, T; X)$ such that u_n is a strong solution of $u_n' + Au_n \ni f_n$ on $[0, T]$ satisfying $u_n(0) = u_{0,n}$, $\lim_{n\to\infty} u_{0,n} = u_0$ in X, $\lim_{n\to\infty} f_n = f$ in $L^1(0, T; X)$, and $\lim_{n\to\infty} u_n = u$ in $\mathscr{C}(0, T; X)$. This notion of solution has been considered e.g. in [3]. If A is the operator constructed in Theorem 2.1, the only weak solution of $u' + Au \ni f$ on $[0, T]$ are the constant solutions. Thus strong solutions, and more generally weak solutions, are not suitable to solve (2) in Banach spaces which fail the Radon-Nikodym property.

5. In order to solve (2) in a general Banach space X, one has to introduce the notion of mild solution: a function $u \in \mathscr{C}(0, T; X)$ is called a mild solution of (2) on $[0, T]$ if for every $\varepsilon > 0$, there exists $n \in \mathbf{N}$, $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n \leqq T$ and $x_0, \dots, x_n, y_1, \dots, y_n \in X$ such that

$$(x_i - x_{i-1})/(t_i - t_{i-1}) + Ax_i \ni y_i \quad \text{for } i = 1, \dots, n,$$

$$|t_i - t_{i-1}| \leqq \varepsilon \quad \text{for } i = 1, \dots, n, \quad \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |f(t) - y_i|\, dt \leqq \varepsilon,$$

$$|u_0 - x_0| \leqq \varepsilon, \qquad |u(t) - x_{i-1}| \leqq \varepsilon \quad \text{for every } t \in [t_{i-1} - t_i],\ i = 1, \dots, n.$$

It is known that (2) has a unique mild solution $u \in \mathscr{C}(0, T; X)$ for every $u_0 \in \overline{D(A)}$ and every $f \in L^1(0, T; X)$. This shows existence and uniqueness

of a solution of certain concrete partial differential equations (for instance, for the Hamilton–Jacobi equations which can be solved in $X = BUC(\mathbf{R}^n)$ or for the nonlinear conservation law equations which can be solved in $X = L^1(\mathbf{R}^N)$).

In a Banach space that has the Radon-Nikodym property, u is a mild solution of (2) on $[0, T]$ if and only if u is a weak solution of (2) on $[0, T]$. In a general Banach space, if u is a weak solution of (2) on $[0, T]$, then u is a mild solution of (2) on $[0, T]$. We refer the reader to [4] for these results. Remark 4, together with the above discussion, shows that in every Banach space which fails the Radon-Nikodym property, there exists an operator A which is m-accretive for some equivalent norm in X, and a mild solution of (2) that is not a weak solution. Observe that in the example constructed in the proof of Theorem 2.1, if we set for $t \in [0, T]$, $f(t) = g(t)x_0 + h(t)$ with $g(t) \in \mathbf{R}$ and $h(t) \in Y$, the mild solution of $u' + Au \ni f$ such that $u(0) = u_0$ is given by $u(t) = v(t)x_0 + \phi(v(t))$ where v satisfies $u(0) = v(0)x_0 + \phi(v(0))$ and $v(t) - v(0) = \int_0^t g(t)\,dt$.

6. The operator B as constructed in Theorem 2.1 is such that B and $-B$ are m-accretive. It is easy to check that if $z = tx_0 + y$ with $t \in \mathbf{R}$ and $y \in Y$, then if $\lambda > 0$:

$$(I + \lambda A)^{-1}(z) = (t + \lambda)x_0 + \phi(t + \lambda)$$

thus for every $n \in \mathbf{N}$:

$$(I + \lambda A/n)^{-n}(z) = (t + \lambda)x_0 + \phi(t + \lambda)$$

and so, by the exponential formula, the mild solution of $u' + Bu \ni 0$ such that $u(0) = 0$ is given by $u(t) = tx_0 + \phi(t)$. Similarly, the mild solution of $v' - Bv \ni 0$ such that $v(0) = Tx_0 + \phi(T)$ is given by $v(t) = (T-t)x_0 + \phi(T-t) = u(T-t)$; thus we have time reversibility for the equation $u' + Bu \ni 0$. This phenomenon is not always true (see e.g. [5]). On the other hand, it is known that if u is a weak solution of $u' + Au \ni 0$ on $[0, T]$ then v defined by $v(t) = u(T - t)$ is a weak solution of $v' - Av \ni 0$ on $[0, T]$. In our example, u and v are mild solutions which are not weak solutions. So time reversibility for the equation $u' + Bu \ni 0$ does not force the solutions to be weak solutions.

3. Appendix

In this section, we construct, for every Banach space X that fails the Radon-Nikodym property, a Lipschitz mapping $\phi : \mathbf{R} \to X$ which is almost nowhere differentiable. The following result is shown in [1]:

Proposition 3.1. *Let X be a Banach space. Then X has the Radon-Nikodym property if and only if every Lipschitz function from $[0, 1]$ into X is differentiable almost everywhere.*

Thus, if X fails the Radon-Nikodym property, there exists a Lipschitz map $\phi : [0, 1] \to X$ such that $\mu(N_\phi) > 0$, where μ denotes the Lebesgue measure on $[0, 1]$ and N_ϕ is the set of points of nondifferentiability of ϕ. In this

section we prove:

Theorem 3.2. *Let X be a Banach space which does not have the Radon-Nikodym property. Then there exists a Lipschitz mapping $\psi : [0, 1] \to X$ such that $\mu(N_\psi) = 1$.*

Proof. Let $Y = \mathrm{Lip}([0, 1], X)$ be the space of all Lipschitz mappings from $[0, 1]$ into X which vanish at zero. For $f \in Y$ let us define

$$\|f\|_Y = \sup\left\{\frac{\|f(y) - f(y)\|}{y - x} : x, y \in [0, 1], x \neq y\right\}.$$

Then $(Y, \| \|_Y)$ is a Banach space. For $n \geqq 1$, let us consider the set

$$U_n = \{f \in Y, \mu(N_f) > 1 - \tfrac{1}{n}\}.$$

U_n is open. Let $f \in U_n$ and $x \in [0, 1]$. Let us denote

$$D_f(x) = \lim_{\varepsilon \to 0} \operatorname{diam}\left\{\frac{f(x+h) - f(x)}{h} : |h| < \varepsilon\right\}.$$

Since $x \in N_f$ if and only if $D_f(x) > 0$, we have

$$N_f = \bigcup\{A_k : k \geqq 1\}, \quad \text{where } A_k = \{x \in [0, 1] : D_f(x) \geqq \tfrac{1}{k}\}.$$

Since (A_k) is increasing, there exists $k \geqq 1$ such that $\mu(A_k) > 1 - 1/n$. We claim that if $u \in Y$ and if $\|u\|_Y < 1/2k$ then $A_k \subseteq N_{f+u}$, thus showing that $f + u \in U_n$ and hence, that U_n is open.

To prove the claim, it is enough to notice that for every $x \in [0, 1]$:

$$\lim_{\varepsilon \to 0} \operatorname{diam}\left\{\frac{u(x+h) - u(x)}{h} : |h| < \varepsilon\right\} < \frac{1}{k};$$

thus, for every $x \in A_k$:

$$\lim_{\varepsilon \to 0} \operatorname{diam}\left\{\frac{(f(x+h) + u(x+h)) - (f(x) + u(x))}{h} : |h| < \varepsilon\right\} > 0.$$

U_n is dense. Let $x \in (0, 1)$ be such that $\mu(I \cap N_\phi)/\mu(I)$ tends to 1 when the diameter of the interval I containing x tends to 0. Let $I = [a, b]$ be an interval contained in $[0, 1]$ and containing x such that $\mu(I \cap N_\phi)/\mu(I) > 1 - 1/2n$. The map ψ defined by $\psi(t) = \phi(a + t(b - a)) - \phi(a)$ belongs to Y and $\mu(N_\psi) > 1 - 1/2n$. Let $f \in Y$. We claim that if $\alpha > 0$ is small enough, then $f + \alpha\psi \in U_n$ which proves that U_n is dense. Indeed, let $\gamma > 0$ sufficiently small to have:

$$\mu(\{x \in N_f : D_f(x) > \gamma\}) > \mu(N_f) - 1/2n.$$

If we take $\alpha = \gamma/\|\psi\|_Y$, we get that:

$$N_{f+\alpha\psi} \supseteq ([0, 1] \setminus N_f) \cap N_\psi = A,$$
$$N_{f+\alpha\psi} \supseteq \{x \in N_f : D_f(x) > \gamma\} = B,$$

and

$$\mu(A \cup B) = \mu(A) + \mu(B) > (1 - \mu(N_f) - \tfrac{1}{2n}) + (\mu(N_f) - \tfrac{1}{2n}) = 1 - \tfrac{1}{n}.$$

Now, by Baire theorem, we have that $G = \bigcap_{n \in \mathbf{N}} U_n$ is a dense $\mathscr{G}_\delta$ in Y and any $\psi \in G$ satisfies Theorem 3.2.

For proving Theorem 2.1, we have used the following consequence of Theorem 3.2.

Corollary 3.3. *Let X be a Banach space which fails the Radon-Nikodym property and Y be a hyperplane of X. Then there exists a Lipschitz mapping $\phi : \mathbf{R} \to Y$ such that the set of all points of differentiability of ϕ is negligible.*

Proof. Since Y fails the Radon-Nikodym property, there exists by Theorem 2.2 a Lipschitz mapping $\psi : [0, 1] \to Y$ such that $\mu(N_\psi) = 1$. Now, if for $x \in \mathbf{R}$ we denote $[x]$ the greatest integer $\leqq x$, the function $\phi : \mathbf{R} \to Y$ defined by $\phi(x) = [x](\psi(1) - \psi(0)) + \psi(x - [x])$ is Lipschitz continuous on $\mathbf{R}$ and satisfies Corollary 3.3.

Remark 3.4. If $X = L^1([0, 1])$ the function $\phi : [0, 1] \to X$ defined by $\phi(t) = \mathbf{1}_{[0,T]}$ where $\mathbf{1}_{[0,T]}$ is the indicator function of $[0, 1]$, is nowhere differentiable. The following improvement of Theorem 3.2 seems to be unknown:

If X is a Banach space which does not have the Radon-Nikodym property, does there exist a Lipschitz map $\phi : [0, 1] \to X$ which is nowhere differentiable?

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