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THE SPACE $(l_\infty/c_0$, WEAK) IS NOT A RADON SPACE

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ABSTRACT. Talagrand [10] gives an example of a Banach space with weak topology which is not a Radon space, independently of their weight. This result gives an answer to a question formulated by Schwartz [9]. In this paper, following the papers of Drewnowski and Roberts [1] and Talagrand [10], we prove that the classical space $(l_\infty/c_0$, weak) is not a Radon space.

Introduction. A Hausdorff topological space E is said to be a Radon space if every finite Borel measure is a Radon measure; i.e.,

$$\mu(A) = \{\mu(K) : K \subset A,\ K \text{ compact}\}$$

for each Borel subset A of E.

We shall say that a cardinal α is of measure zero (resp. nonmeasurable) if there is not a real-valued, diffuse, nontrivial measure (resp. $\{0, 1\}$-valued), on the power set of a set A with cardinal α.

The weight (density character) of a topological space E is the smallest cardinal such that there exists in E a dense subset A with this cardinal.

A topological space E has the α-property of Lindelöf, α a transfinite cardinal, if for each family $(G_i)_{i\in I}$ of open subsets of E there exists $J \subset I$ such that $\operatorname{card}(J) \leq \alpha$ and $\bigcup_{i\in I} G_i = \bigcup_{i\in J} G_i$. The smallest cardinal α such that E has the α-property of Lindelöf is called the L-weight of E.

Likewise, E is a Flock space if for every well-ordered family $(G_i)_{i\in I}$ of open sets, with $H_\alpha = G_\alpha \setminus \bigcup_{\beta<\alpha} G_\beta$, every union $\bigcup_{\alpha\in J} H_\alpha$ $(J \subseteq I)$ is universally Borel-measurable. This property is an extrapolation of the Montgomery lemma [7], which proves that every metrizable space is Flock. Also, the strongly Lindelöf spaces are Flock spaces.

Let I be a noncountable set and $x \notin I$. Consider the topology T on $I \cup \{x\}$ such that the discrete topology is on I and the neighborhoods of x are the complementary sets of countable subsets of I. Let K be the Stone-Čech compactification of this space. Then, as Talagrand [10] proved, $(C(K)$, weak) is not a Radon space.

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Talagrand also proved that $(Z$, weak) is not a Radon space when Z is the subspace of $l^\infty(I)$ of the countable support functions and $\operatorname{card}(I) = \aleph_1$. Following that proof and the work of Drewnowski and Roberts [1], this paper is devoted to proving that the classical space $(l_\infty/c_0$, weak) is not a Radon space. Note that this fact could be proved if Z were a subspace of l_∞/c_0, but this is not yet known. Let us remark that it is easily proved that $(E$, weak) is not a Radon space if the weight of E is real-measurable.

It is easily deduced that every Borel measure on $(l_\infty$, weak) valued on $\{0, 1\}$ is a δ_x.

In Theorem 18 [2] we proved that a Banach space with the weak topology is a Radon space if and only if it is Flock and its L-weight is of measure zero. Thus, in the last part of this paper, we prove that l_∞/c_0 is not a Flock space. Conversely, in [4], we improved Schachermayer's theorem, proving that a subset Ω of a WCG Banach space with the weak topology whose weight is of measure zero is a Radon space of type $(\mathscr{F})$; that is, every finite Borel measure on $(\Omega$, weak) is τ-additive.

1. **Theorem.** *On $(l_\infty/c_0$, weak) there exists a Borel measure $\mu \neq 0$ valued on $\{0, 1\}$ such that the space is not a Radon measure.*

Proof. Let $\beta\omega$ be the Stone-Čech compactification of $\omega = \mathbb{N}$ with discrete topology. Let $\mathscr{A}$ be the clopen algebra of $K = \beta\omega\backslash\omega$ and $\mathscr{A}_0 = \mathscr{A}\backslash\{\varnothing\}$. $\mathscr{A}$ has the following property (called the Cantor separability property in [11]): For every decreasing sequence (A_n) in $\mathscr{A}_0$ there exists an $A \in \mathscr{A}_0$ such that $A \subset \bigcap_1^\infty A_n$.

Let $\mathscr{V} = (V_\alpha)_{\alpha\in A}$, a maximal well-ordered family in $\mathscr{A}_0$. Then every countable intersection $\bigcap_1^\infty V_{\alpha_n}$ contains a $V \in \mathscr{V}$. We denote by F the set of continuous functions from K in $\{0, 1\}$ which vanish in $H = \bigcap\{V : V \in \mathscr{V}\}$.

Like Talagrand, we are going to make two classes $\mathscr{C}$ and $\mathscr{D}$ of weak-Borel sets (Borel with the weak topology) in $F \subset C(K)$ such that

(i) the smallest σ-algebra containing $\mathscr{C}$ is the class $\mathscr{B}$ of the weak-Borel sets of F;
(ii) if $C \in \mathscr{C}$, then either $C \in \mathscr{D}$ or $F\backslash C \in \mathscr{D}$;
(iii) every countable intersection of elements of $\mathscr{D}$ is not empty; and
(iv) for every $t \in K\backslash H$, $\{f \in F : f(t) = 1\} \in \mathscr{D}$.

Now, we define a Borel measure μ on $(F$, weak) by $\mu(B) = 1$ if $B \in \mathscr{B}$ and contains a countable intersection of members of $\mathscr{D}$, and $\mu(B) = 0$ in the other case. The measure μ is not τ-additive, because F is the union of the open sets $G_t = \{f \in F : f(t) = 0\}$ when $t \in K\backslash H$, $\mu(F) = 1$, and each $\mu(G_t) = 0$. It follows that there exists a Borel measure μ_0 on $(C(K)$, weak) $\approx$ $(l_\infty/c_0$, weak) (see [6]) valued in $\{0, 1\}$ which is not a Radon measure.

Let k be an integer, J a set, $(\mu_j^p)_{j\in J, p\leq k}$, of Radon measures on K, and $(a^p)_{p\leq k}$, $(b^p)_{p\leq k}$ rational numbers such that $a^p < b^p$. By definition, $\mathscr{C}$ is the

class of sets $C = \bigcup_{j\in J} U_j$, where

$$U_j = \{f \in F : \forall p \le k,\ \mu_j^p(f) \in (a^p, b^p)\},$$

varying over all k, J and measures μ_j^p.

The class $\mathscr{D}$ is built up by the sets $C \in \mathscr{C}$ such that, for every $V \in \mathscr{V}$, there exist $j \in J$ and $f \in U_j$ with $f = 1$ in $K\backslash V$, and with the complementaries $F\backslash C$ of the sets $C \in \mathscr{C}$ such that they do not have this condition. Only (iii) has to be proved. To do this, it is enough that, for every sequence $(C_n) \subset \mathscr{C}\cap\mathscr{D}$ and every $V_0 \in \mathscr{V}$, there exists $f \in \bigcap_n C_n$ such that $f = 1$ in $K\backslash V_0$. It is easily proved (with the evident notations) that for every $n \in \mathbb{N}$, there exists $\varepsilon_n > 0$ such that, if $W \in \mathscr{V}$, then there exist $j \in J_n$ and $f \in U_j'$ with $f = 1$ on $K\backslash W$, where

$$U_j' = \{f \in F : \forall p \le k_n,\ \mu_{j,n}^p(f) \in (a_n^p + \varepsilon_n, b_n^p - \varepsilon_n)\}.$$

Let $\mathscr{U}_n$ be an ultrafilter on J_n which contains all sets

$$\{j \in J_n : \exists f \in U_j',\ f = 1 \text{ on } K\backslash V\}$$

when $V \in \mathscr{V}$. Given $p \le k_n$, let $\upsilon_n^p = \lim_{\mathscr{U}_n} \mu_{j,n}^p$. Then there exists $V \in \mathscr{V}$, $V \subset V_0$, such that $\upsilon_n^p(V\backslash W) = 0$ for every $W \in \mathscr{V}$ and $p \le k_n$, $n \in \mathbb{N}$. Similarly, by means of an easy induction, there exist $f_n \in F$, $j_n \in J_n$, V_{α_n}, $V_{\alpha_n}' \in \mathscr{V}$ and clopen sets H_{n0}, H_{n1} in $\beta\omega$ such that

(i) $\mu_{j_n,n}^p(V\backslash V_{\alpha_n}) \le \varepsilon_n/2$, for every n and $p \le k_n$;

(ii) $f_n \in U_{j_n}'$ and $f_n = 1$ on $K\backslash V_{\alpha_n}$;

(iii) $f_n = 0$ on $V_{\alpha_n}' \subset V_{\alpha_n}$ and $\mu_{j_n,n}^p(V_{\alpha_n}'\backslash H) = 0$ for every n and $p \le k_n$;

(iv)

$$A_{00} = \varnothing, \qquad A_{01} = K\backslash V,$$
$$A_{n0} = \{t \in V_{\alpha_n}\backslash V_{\alpha_n}' : f_n(t) = 0\},$$
$$A_{n1} = \{t \in V_{\alpha_n}\backslash V_{\alpha_n}' : f_n(t) = 1\};$$

(v) H, H_{n0}, H_{n1} are pairwise disjoint sets such that $H_{ni} \supset A_{ni} \cup H_{n-1,i}$, $(H_{0i} = A_{0i})$ for $i = 0, 1$;

(vi) The set $V_{\alpha_{n+1}}$ verifies that $V_{\alpha_{n+1}} \subset V_{\alpha_n}'$ and $V_{\alpha_{n+1}} \cap (H_{n0} \cup H_{n1}) = \varnothing$.

Clearly the open sets $G_0 = \bigcup_{n=1}^\infty H_{n0}$ and $G_1 = \bigcup_{n=1}^\infty H_{n1}$ are disjoint, but $\overline{G_0}\cap\overline{G_1} = \varnothing$ also is, because if $t \in \overline{G_0}\cap\overline{G_1}$, then $t \in \beta\omega$ would be a cluster point of two disjoint sets of integer numbers contained in G_0 and G_1, respectively. As $(\overline{G_0}\cup\overline{G_1})\cap H = \varnothing$, then there exists a clopen H_1 in K such that $\overline{G}_1 \cap K \subset H_1$, $\overline{G}_0 \cap K \cap H_1 = \varnothing$, and $H \cap H_1 = \varnothing$.

Function $f = \chi_{H_1} \in F$ satisfies $f = f_n$ on $A_n = A_{n0} \cup A_{n1}$. Moreover, since for every n and every $p \le k_n$, we have that $\mu_{j_n}^p(f \ne f_n) < \varepsilon_n$. It follows that $f \in U_{j_n}$ for every n. Then $f \in \bigcap_{n=1}^\infty C_n$ and $f = 1$ on $K\backslash V_0$.

2. **Corollary.** $(l_\infty/c_0$, weak) *is not a Radon space.*

3. **Theorem.** *Let Ω be a Flock space whose L-weight is nonmeasurable. Then every finite perfect Borel measure μ on Ω is τ-additive.*

Proof. It is enough to prove that, for every family of open sets $(G_\alpha)_{\alpha\in A}$ in Ω,

$$\mu\left(\bigcup_{\alpha\in A} G_\alpha\right) = \sup_J \mu\left(\bigcup_{\alpha\in J} G_\alpha\right),$$

where the supremum is taken over all finite subsets J in A.

From Zermelo's theorem, and since the L-weight of Ω is nonmeasurable, then we can suppose A is well ordered and cardinally nonmeasurable. Let $H_\alpha = G_\alpha \setminus \bigcup_{\beta<\alpha} G_\beta$. Since Ω is a Flock space, the union $\bigcup_{\alpha\in A}$, H_α of every subfamily of $(H_\alpha)_{\alpha\in A}$ is a universally Borel-measurable set. By [5, Theorem 2.5], if $S = \{\alpha \in A : \mu(H_\alpha) \neq 0\}$, then S is countable and $\mu(\bigcup_{\alpha\notin S} H_\alpha) = 0$.

Now,

$$\mu\left(\bigcup_{\alpha\in A} G_\alpha\right) = \mu\left(\bigcup_{\alpha\in A} H_\alpha\right) = \mu\left(\bigcup_{\alpha\in S} H_\alpha\right) \le \mu\left(\bigcup_{\alpha\in S} G_\alpha\right) \le \sup_J \mu\left(\bigcup_{\alpha\in J} G_\alpha\right),$$

so $\mu(\bigcup_{\alpha\in A} G_\alpha) = \sup_J \mu(\bigcup_{\alpha\in J} G_\alpha)$.

4. **Corollary.** $(l_\infty/c_0$, weak) *is not a Flock space.*

Proof. Follows from Theorems 1 and 3, since cardinal of l_∞/c_0 is nonmeasurable.

Remark. Since [3, Theorem 6] says that every Radon space is a Flock space, Corollary 4 improves Corollary 2. Even this theorem proves that a regular space E is Flock and its weight is of measure zero if and only if E is a Radon space of type $(\mathscr{F})$.

5. **Theorem.** *If (φ_n) is a sequence of continuous functions which separates points on a topological space E, then every Borel measure $\mu \neq 0$ on E which takes values in $\{0, 1\}$ is concentrated in a point.*

Proof. Let υ be the Borel measure on $\mathbb{R}$ defined by

$$\upsilon(B) = \mu(\varphi_1^{-1}(B))$$

on the class $\mathscr{B}$ of Borel measure on $\mathbb{R}$. Then, as υ takes only the values 0 and 1, there exists $\alpha_1 \in \mathbb{R}$ such that $\mu(\varphi_1^{-1}(\alpha_1)) = 1$. Now, by induction, we can construct a sequence (α_n) in $\mathbb{R}$ such that

$$\mu\left(\bigcap_{k=1}^{n} \varphi_k^{-1}(\alpha_k)\right) = 1$$

for every n. To do this, it is enough to consider the following measure:

$$\upsilon(B) = \mu\left(\bigcap_{k=1}^{n} \varphi_k^{-1}(\alpha_k) \cap \varphi_{n+1}^{-1}(B)\right) \qquad (B \in \mathscr{B}).$$

Then

$$\mu\left(\bigcap_{n=1}^{\infty} \varphi_n^{-1}(\alpha_n)\right) = 1 .$$

Finally, as (φ_n) separates points of E, then $\bigcap_{n=1}^{\infty} \varphi_n^{-1}(\alpha_n)$ contains only a point x. So μ is concentrated in x.

6. **Corollary.** *Every Borel measure* $\mu \neq 0$ *on* $(l_\infty$, weak) *taking values in* $\{0, 1\}$ *is concentrated in a point.*

This allows us to formulate the following open question: Is $(l_\infty$, weak) a Flock space?

Remark. Drewnowski and Roberts [1] have proved that for every Schauder decomposition (finite or infinite) $C = l_\infty/c_0 = X_1 + X_2 + \cdots$, at least one of X_n contains a space isomorphic to C, complemented in C. It can easily be proved, as in Theorem 5, that if C is a Banach space with a Schauder decomposition and there exists a nontrivial Borel measure μ on C valued in $\{0, 1\}$, then there exist $n \in \mathbb{N}$ and a nontrivial Borel measure μ_n on X_n taking values in $\{0, 1\}$. So, taking into account that $C \approx C \oplus C$ and $C \approx l_\infty \oplus C$ $(C = l_\infty/c_0)$ the following question naturally appears:

Open question. If X is a complemented subspace in l_∞/c_0 on which there exists a nontrivial measure taking values in $\{0, 1\}$, then does X have a complemented copy of C?

Let $L = L_\infty([0, 1]^{\mathfrak{c}})$, where $\mathfrak{c} = 2^{\aleph_0}$ and $[0, 1]$ is considered with the product Lebesgue measure. If we suppose $\mathfrak{c} = \aleph_1 (CH)$, then L is isometric to a complemented and closed subspace X of C.

Another question arises:

Open question. Does there exist on L a nontrivial Borel measure μ which only takes $0, 1$ values?

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