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SPECTRAL PROPERTIES OF PERTURBED LINEAR OPERATORS AND THEIR APPLICATION TO INFINITE MATRICES

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ABSTRACT. In this paper, upper bounds for the difference between the eigenvalues and the eigenvectors of a closed linear operator D and those of $D+F$, where F is a bounded linear operator, are given in terms of the norm of F. These results are applied to approximate the eigenvalues and the eigenvectors of a diagonally infinite matrix by those of its corresponding diagonal matrix.

1. INTRODUCTION

Let T be a closed operator in a Banach space $\mathscr{X}$ and F be an operator on $\mathscr{X}$. One of the basic problems in perturbation theory for closed operators is to study the relation between the eigenvalues and the eigenvectors of T and those of $T+F$. The operator F is called the perturbation.

To the author's knowledge, the first explicit quantitative results regarding this problem were first published by Rosenbloom [5]. In [5], upper bounds for the norm of the perturbation are given to guarantee that the eigenvalues and the eigenvectors of the perturbed linear operator stay close to those of the unperturbed one. These results were established by the application of an elementary fixed point theorem (see [5, Lemma 1]).

In §2 we introduce the main theorem (Theorem 2.1) that improves the results of [5] (see Remark 2.3). The proof of the main theorem is based on the contraction mapping theorem [4, Theorem 1, p. 474].

In §3, we apply the results of §2 along with [2, Theorem 4.2] to the case of a diagonally dominant infinite matrix acting as a linear operator in the sequence space l_2, where it is established that the eigenvalues of such a matrix approach its main diagonal. The results of §3 are used to approximate the eigenvalues of Mathieu's equation in §4.

Before stating our first theorem, we introduce some notation that will be used throughout the paper.

The sets of positive integers and complex numbers are denoted by $\mathbb{N}$ and $\mathbb{C}$, respectively.

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The domain of an operator A in a Banach space $\mathscr{X}$ is denoted by $\mathscr{D}(A)$. For a matrix $A = (a_{ij})$ acting in l_p, $1 \le p \le \infty$, the symbol $\sum_j'$ will always denote the sum from one to infinity excluding the index $j = i$. We define the row and column sums of A:

$$P_i = \sum_j{}' |a_{ij}|, \qquad Q_i = \sum_j{}' |a_{ji}|,$$

and the corresponding Gersgorin discs (where they exist):

$$\mathscr{R}_i = \{z \in \mathbb{C} : |z - a_{ii}| \le P_i\}, \qquad \mathscr{C}_i = \{z \in \mathbb{C} : |z - a_{ii}| \le Q_i\}.$$

The algebra of bonded linear operators A on a Banach space $\mathscr{X}$ with domains $\mathscr{D}(A) = \mathscr{X}$ is denoted by $L(\mathscr{X})$. The identity operator on $\mathscr{X}$ is denoted by I.

2. The main theorem

In this section we show under certain conditions (see Theorem 2.1) that if the unperturbed system has an eigenvalue λ_0 with a corresponding eigenvector x_0, then an upper bound for the perturbation (in norm) can be given to guarantee that the perturbed system has an eigenvalue with a corresponding eigenvector that are both close to λ_0 and x_0, respectively.

Theorem 2.1. *Let D be a closed operator defined in a Banach space $\mathscr{X}$ such that $\mathscr{D}(D)$ is dense in $\mathscr{X}$ and assume that*

(1) *The point λ_0 is an eigenvalue of both D and D' (the dual of D) with corresponding eigenvectors x_0 and x_0', respectively. The vector x_0 satisfies $\|x_0\| = 1$.*
(2) *The restriction of the operator $(D - \lambda_0 I)$ to the space $\mathscr{X}_1 = \{x \in \mathscr{X} : x_0'(x) = 0\}$ has a bounded inverse R mapping $\mathscr{X}_1$ into itself.*
(3) *The vector x_0' satisfies $x_0'(x_0) = 1$.*

Then for any $r \in (0, (\|S\|/(\|x_0'\|\,\|R\|))^{1/2})$ and $F \in \mathscr{G}_r = \{U \in L(\mathscr{X}) : \|U\| \le \delta\}$, where

$$\delta = r/(\|x_0'\|\,\|R\| r^2 + (\|x_0'\|\,\|R\| + \|S\|)r + \|S\|)$$

and

$$S = R(I - x_0 x_0'),$$

the system of equations

$$Ax = (D + F)x = \lambda x, \qquad x_0'(x) = 1 \tag{2.1}$$

has the unique solution (x, λ) in the set $\{z \in \mathscr{D}(D) : \|z - x_0\| \le r\} \times \mathbb{C}$. (Concerning the location of λ in $\mathbb{C}$, see Remark 2.2.)

Proof. Define $P = I - x_0 x_0'$. Hence $S = RP$.

Fix $r \in (0, (\|S\|/(\|x_0'\| \|R\|))^{1/2})$ and $F \in \mathscr{G}_r$. Writing $x = x_0 + y$ and $\lambda = \lambda_0 + \eta$, and substituting into equation (2.1), we find it is required to solve the equation:

$$(D - \lambda_0 I)y = \eta(x_0 + y) - F(x_0 + y), \tag{2.2}$$

where $y \in \mathscr{Y} = \mathscr{X}_1 \cap \{z \in \mathscr{X} : \|z\| \le r\} \cap \mathscr{D}(D)$. From the second hypothesis, the left-hand side of equation (2.2) is in $\mathscr{X}_1$, and so acting on both sides of the equation by x_0' we get

$$\eta = \eta x_0'(x_0 + y) = x_0'(F(x_0 + y)). \tag{2.3}$$

Therefore it is required to solve in $\mathscr{Y}$ the equation

$$(D - \lambda_0 I)y = x_0'(F(x_0 + y))(x_0 + y) - F(x_0 + y). \tag{2.4}$$

From the definition of P, it is easy to see that solving equation (2.4) in $\mathscr{Y}$ is equivalent to solving the equation:

$$(D - \lambda_0 I)y = x_0'(F(x_0 + y))y - PF(x_0 + y) \tag{2.5}$$

in $\mathscr{Y}$. Now we may use hypothesis (2) and act on both sides of equation (2.5) by the operator R to deduce that the equation

$$y = x_0'(F(x_0 + y))Ry - SF(x_0 + y) \tag{2.6}$$

has the same solution set in $\mathscr{Y}$ as does equation (2.5).

Define the map Q_F on the closed set $\mathscr{Y}_1 = \mathscr{X}_1 \cap \{z \in \mathscr{X} : \|z\| \le r\}$ by the equation:

$$Q_F(y) = x_0'(F(x_0 + y))Ry - SF(x_0 + y)$$

for all $y \in \mathscr{Y}_1$. We prove that Q_F is a contraction map mapping $\mathscr{Y}_1$ into itself.

Let $y \in \mathscr{Y}_1$. Since P maps $\mathscr{X}$ onto $\mathscr{X}_1$ (for $x \in \mathscr{X}$, we have $x_0'(Px) = x_0'(x) - x_0'(x)x_0'(x_0) = x_0'(x) - x_0'(x) = 0$, by the third hypothesis), we have $PF(x_0 + y) \in \mathscr{X}_1$. Hence from the second hypothesis, both $SF(x_0 + y)$ and Ry are in $\mathscr{X}_1$. Thus $Q_F(y) \in \mathscr{X}_1$. Also using the triangular inequality and the fact that $\|x_0\| = 1$ and $\|y\| \le r$, we get

$$\|Q_F(y)\| \le (\|x_0'\| \|F\| \|R\| r + \|S\| \|F\|)(1 + r).$$

But since $\|F\| \le r/(\|x_0'\| \|R\| r^2 + (\|x_0'\| \|R\| + \|S\|)r + \|S\|)$, we have $\|Q_F(y)\| \le r$, and this proves that Q_F maps $\mathscr{Y}_1$ into itself. Now let y_1 and y_2 be in $\mathscr{Y}_1$. We have

$$\begin{aligned}\|Q_F(y_1) - Q_F(y_2)\| &\le \|F\|(2\|x_0'\| \|R\| r + \|x_0'\| \|R\| + \|S\|)\|y_1 - y_2\| \\ &\le \frac{2\|x_0'\| \|R\| r + \|x_0'\| \|R\| + \|S\|}{\|r_0'\| \|R\| r + \|x_0'\| \|R\| + \|S\| + \|S\|/r}\|y_1 - y_2\|.\end{aligned}$$

Hence we have

$$\|Q_F(y_1) - Q_F(y_2)\| \le \alpha\|y_1 - y_2\|,$$

$$\text{where } \alpha = \frac{2\|x_0'\| \|R\| r + \|x_0'\| \|R\| + \|S\|}{\|x_0'\| \|R\| r + \|x_0'\| \|R\| + \|S\| + \|S\|/r}.$$

But since

$$0 < r < (\|S\|/(\|x_0'\| \, \|R\|))^{1/2}$$

then $\alpha \in (0, 1)$ and this completes the proof that Q_F is a contraction map mapping $\mathscr{Y}_1$ into itself. Thus from [4, Theorem 1, p. 474], Q_F has the unique fixed point $y^* = Q_F(y^*) \in \mathscr{Y}_1$, and from hypothesis (2) and the definition of Q_F, $y^* \in \mathscr{Y}$. Hence equation (2.6) has the unique solution y^* in $\mathscr{Y}$. But since the two equations (2.4) and (2.6) have the same solution set in $\mathscr{Y}$, then equation (2.2) has a unique solution (y^*, η^*) in $\mathscr{Y} \times \mathbb{C}$ where y^* is the unique solution of equation (2.4) and η^* is given by the equation:

$$\eta^* = x_0'(F(x_0 + y^*)).$$

Thus the system (2.1) has the unique solution $(x, \lambda) \in \{z \in \mathbb{C} : \|z - x_0\| \leq r\} \times \mathbb{C}$, where $x = x_0 + y^*$ and $\lambda = \lambda_0 + \eta^*$. This completes the proof of the theorem.

Remark 2.2. In Theorem 2.1, by choosing r and δ small enough, the solution (x, λ) of the system (2.1) can come close to x_0, λ_0 to any degree of accuracy. To explain this point, let $\varepsilon > 0$. If we let r vary in the interval $(0, (\|S\|/(\|x_0'\| \, \|R\|))^{1/2})$, then

$$\delta = \delta(r) = r/(\|x_0'\| \, \|S\| r^2 + (\|x_0'\| \, \|R\| r + \|S\|) r + \|S\|)$$

is a function of r. This function has a greatest lower bound equal to zero, and is increasing since the derivative:

$$\delta'(r) = \frac{\|S\| - \|x_0'\| \, \|R\| r^2}{(\|x_0'\| \, \|R\| r^2 + (\|x_0'\| \, \|R\| + \|S\|) r + \|S\|)^2} > 0$$

for all $r \in (0, (\|S\|/(\|x_0'\| \, \|R\|))^{1/2})$. So if we choose $r < \varepsilon$ such that

$$\delta = \delta(r) < \varepsilon/(\|x_0'\|(1 + r)),$$

the unique solution (x, λ) of the system (2.1) in the product space $\{z \in \mathscr{D}(D) : \|z - x_0\| \leq r\} \times \mathbb{C}$ satisfies

$$\|x - x_0\| < \varepsilon$$

and

$$|\lambda - \lambda_0| \leq \|x_0'\|(1 + \|y\|)\|F\| < \frac{\varepsilon \|x_0'\|(1 + \|y\|)}{\|x_0'\|(1 + r)} \leq \varepsilon.$$

Remark 2.3. In [5] it is proved that if the hypotheses of Theorem 2.1 hold, $r > 0$ and

$$\|F\| \leq \psi(r) = r/(2\|x_0'\| \, \|R\| r^2 + (\|x_0'\| \, \|R\| + \|S\|) r + \|S\|)$$

then the conclusion of the theorem follows. Although the domain of the function $\delta(r)$ in Theorem 2.1 is restricted to the interval $(0, r_0)$, where $r_0 = (\|S\|/(\|x_0'\| \, \|R\|))^{1/2}$, we can prove that given $r_1 \in (0, \infty)$, there exists $r_2 \in (0, r_0)$ such that $r_2 \leq r_1$ and $\delta(r_2) > \psi(r_1)$. This assertion is proved by

considering the cases $r_1 < r_0$ and $r_1 < r_0$ separately.

(i) If $r_1 < r_0$, then from the definitions of $\delta(r)$ and $\psi(r_1)$, it is clear that $\delta(r_1) > \psi(r_1)$.

(ii) If $r_1 \geq r_0$, then since $\psi(r)$ attains its maximum at

$$r_0^* = (\|S\|/(2\|x_0'\| \, \|R\|))^{1/2} < r_0$$

and $\psi(r)$ is continuous, there exists $r_2 < r_0^*$ such that $\psi(r_2) = \psi(r_1)$. So from case (i), $\delta(r_2) > \psi(r_2) = \psi(r_1)$.

3. Diagonally dominant infinite matrices

In this section we show that for a diagonally dominant infinite matrix operator acting in l_2 and having a discrete spectrum consisting of simple eigenvalues that diverge to infinity, the eigenvalues of A approach the centers of the Gersgorin discs and their corresponding eigenvectors approach the unit coordinate vectors e_i, $i \in \mathbb{N}$.

In applying Theorem 2.1 to an infinite matrix operator $A = (a_{ij})$ acting in l_p, $1 \leq p < \infty$, we consider the diagonal matrix $d = \operatorname{diag}(a_{11}, a_{22}, \dots)$ as the unperturbed operator and regard $F = ((1 - \delta_{ij})a_{ij})$, where δ_{ij} denotes the Kronecker delta, as the perturbation. Since $e_i \in \mathscr{D}(D)$ for all $i \in \mathbb{N}$, then $\mathscr{D}(D)$ is dense in l_p. Hence from [7, Theorem VII.1.1], the dual D' of D exists. Also D' has the same eigenvalues $a_{11}, a_{22}, \dots$ as D has. This follows from the following theorem.

Theorem 3.1. *The dual D' of the matrix operator $D = \operatorname{diag}(a_{11}, a_{22}, \dots)$ in l_p, $1 \leq p < \infty$, exists and has the same eigenvalues a_{ii}, $i \in \mathbb{N}$, as D has.*

Proof. As shown above, the dual D' of D exists.

Let λ be an eigenvalue of D' with a corresponding eigenvector y'. Hence $0 \neq y' \in l_p'$ and $D'y' = \lambda y'$. We have the equation:

$$\lambda y'(x) = y'(Dx) \tag{3.1}$$

for all $x = (x_1, x_2, \dots) \in \mathscr{D}(D)$.

Let $y = (y_1, y_2, \dots)$ be the vector in l_q $(1/p + 1/q = 1)$ which satisfies the equation:

$$y'(x) = \sum_{i=1}^{\infty} x_i y_i \tag{3.2}$$

for all $x = (x_1, x_2, \dots) \in l_p$ (see [6, Theorem III.5.2]). The vector $y \neq 0$ since $\|y\| = \|y'\|$ and $y' \neq 0$. Substituting $x = e_i$, $i \in \mathbb{N}$, in equations (3.1) and (3.2), we get the equation:

$$\lambda y_i = a_{ii} y_i \tag{3.3}$$

for all $i \in \mathbb{N}$.

Since $y \neq 0$, then from equation (3.3) it follows that there is only one nonzero component of the vector y; denote it by y_j. At this component y_j we have $\lambda = a_{jj}$. Thus λ is an eigenvalue of D.

If $\lambda = d_{ii}$, $i \in \mathbb{N}$, then the linear functional $y' \in l_p'$ defined by the equation:

$$y'(x) = x_i,$$

for all $x = (x_1, x_2, \dots) \in l_p$, satisfies

$$(D'y')(x) = y'(Dx) = a_{ii}x_i = \lambda x_i = \lambda y'(x)$$

for all $x = (x_1, x_2, \dots) \in \mathscr{D}(D)$. This proves the $D'y' = \lambda y'$. Since $y' \neq 0$ ($\|y'\| = 1$) then $\lambda = a_{ii}$ is an eigenvalue of D' and this completes the proof of the theorem.

From Theorem 2.1 and Theorem 4.2 of [2], we have

Theorem 3.2. *Let $A = (a_{ij})$ be a matrix operator in l_2 and assume that*

(1) *$\{a_{ii}\}$ is a strictly increasing sequence of real numbers diverging to infinity.*
(2) *For every $i \in \mathbb{N}$,*

$$P_i = \sum_j{}' |a_{ij}| = \sigma_i |a_{ii}|, \quad \textit{where } \sigma_i \in [0, 1].$$

(3) $F = ((1 - \delta_{ij})a_{ij}) \in L(l_2)$.
(4) $|a_{ii} - a_{jj}| > P_i + P_j$ *for all* $i, j \in \mathbb{N}$, $i \neq j$.

Then the spectrum of A consists of a discrete, countable set of eigenvalues $\{\lambda_i : i \in \mathbb{N}\}$ and for every $i \in \mathbb{N}$, λ_i is a simple eigenvalue lying in the set $\mathscr{R}_i$. Furthermore, if $a_{i+1,i+1} - a_{ii} \to \infty$ as $i \to \infty$ then $|\lambda_i - a_{ii}| \to 0$ as $i \to \infty$, and for every $i \in \mathbb{N}$ there exists an eigenvector z_i of A corresponding to the eigenvalue λ_i such that $\|z_i - e_i\| \to 0$ as $i \to \infty$.

Proof. Let $k = \|F\|$ and $k_1 = k + 1$. Since $\|(D + k_1 I)^{-1}\| \leq 1/k_1$, then $\|F(D + k_1 I)^{-1}\| < 1$. Thus the operator $A_1 = (a_{ij}^{(1)}) = A + k_1 I$ satisfies the hypotheses of [2, Theorem 4.2] (in the case $p = 2$). So the spectrum of A_1 consists of a discrete, countable set of eigenvalues $\{\mu_i : i \in \mathbb{N}\}$ and for every $i \in \mathbb{N}$, μ_i is a simple eigenvalue of A_1 lying in the set $\{z \in \mathbb{C} : |z - a_{ii}^{(1)}| \leq P_i\}$. Since $\lambda_i = \mu_i - k_1$, $i \in \mathbb{N}$, there are eigenvalues of A, each of algebraic multiplicity equal to one, and a point μ is in the spectrum of A_1 if and only if $\mu - k_1$ is in the spectrum of A, then the spectrum of A is a discrete, countable set of eigenvalues $\{\lambda_i : i \in \mathbb{N}\}$, and for every $i \in \mathbb{N}$, λ_i is a simple eigenvalue lying in the set $\mathscr{R}_i$. This establishes the first part of the theorem. Now suppose $a_{i+1,i+1} - a_{ii} \to \infty$ as $i \to \infty$. We show that the matrix operator $D = \operatorname{diag}(a_{11}, a_{22}, \dots)$ in l_2 satisfies the hypotheses of Theorem 2.1. It is clear that a_{ii}, $i \in \mathbb{N}$, are the eigenvalues of D with corresponding eigenvectors e_i, $i \in \mathbb{N}$. From Theorem 3.1, it follows that the dual D' of D has the same

eigenvalues as does D and for every $i \in \mathbb{N}$, the linear functional $x_i' \in l_2'$ defined by the equation:

$$x_i'(x) = x_i,$$

for all $x = (x_1, x_2, \dots) \in l_2$, is an eigenvector of D' corresponding to the eigenvalue a_{ii}.

Now fix $i \in \mathbb{N}$ and let $\mathscr{X}_i = \{x \in l_2 : x_i'(x) = 0\}$. It is clear that $e_j \in \mathscr{X}_i$ for all $j \neq i$ and $x_i'(e_i) = 1$. Hence from the continuity of x_i', $\mathscr{X}_i$ is the closure of the set $\operatorname{span}\{e_j : j \in \mathbb{N} - \{i\}\}$. Also it is clear that the matrix operator $R_i = (r_{jk}^{(i)})$ defined by

$$r_{jk}^{(i)} = \begin{cases} c & \text{if } k = j = i, \\ (a_{jj} - a_{ii})^{-1} & \text{if } k = j \neq i, \\ 0 & \text{otherwise}, \end{cases} \tag{3.4}$$

where c is an arbitrary complex number, maps $\mathscr{X}_i$ into itself and is the inverse of the restriction of the operator $(D - a_{ii}I)$ to the space $\mathscr{X}_i$. From equation (3.4), R_i is bounded with norm:

$$\|R_i\| = \max\left\{\frac{1}{a_{ii} - a_{i-1,i-1}}, \frac{1}{a_{i+1,i+1} - a_{ii}}\right\} \tag{3.5}$$

if $i \geq 2$ (if $i = 1$, then $\|R_1\| = (a_{22} - a_{11})^{-1}$). Hence D satisfies all the hypotheses of Theorem 2.1.

Now we show that for all $i \in \mathbb{N}$, the operator $S_i = R_i(I - e_i x_i')$ has the same norm as R_i (from the proof of Theorem 2.1, the operator $I - e_i x_i'$ maps l_2 onto $\mathscr{X}_i$. Hence S_i is well defined with domain l_2). Since $S_i x = R_i x$ for all $x \in \mathscr{X}_i$, then

$$\|R_i\| \leq \|S_i\|. \tag{3.6}$$

On the other hand let $x = \sum_{j=1}^{\infty} \alpha_j e_j \in l_2$ and $\|x\| \leq 1$. We have

$$\begin{aligned}
(I - e_i x_i')(x) &= x - x_i'\left(\sum_{j=1}^{\infty} \alpha_j e_j\right) e_i \\
&= x - x_i'\left(\sum_{j=1}^{\infty} \alpha_j e_j\right) e_i \\
&= x - x_i'\left(\sum_{j=1}^{\infty} \alpha_j x_i'\right) e_i \\
&= x - \alpha_i e_i \\
&= \sum_{j=1}^{\infty} \alpha_j e_j - \alpha_i e_i
\end{aligned}$$

which is an element in $\mathscr{X}_i$; denote it by x. From [3, Theorem I.4.1] we have $\|x\| \leq 1$. Therefore $\|S_i x\| = \|R_i x\| \leq \|R_i\|$. Hence we have

$$\|S_i\| \leq \|R_i\|. \tag{3.7}$$

Equations (3.6) and (3.7) prove $\|S_i\| = \|R_i\|$.

From the hypothesis $a_{i+1,i+1} - a_{ii} \to \infty$ and equation (3.5), it follows that $\|R_i\| \to 0$ as $i \to \infty$. Hence there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$,

$$k\|R_i\| < 1/4. \tag{3.8}$$

For all $i \geq i_0$, we solve (in r_i):

$$k = r_i/((r_i^2 + 2r_i + 1)\|R_i\|). \tag{3.9}$$

The solution of equation (3.9) is given by

$$r_i = \frac{(1 - 2k\|R_i\|) \pm \sqrt{1 - 4k\|R_i\|}}{2k\|R_i\|}. \tag{3.10}$$

From equation (3.8), both values of r_i in equation (3.10) are positive results and the smaller value, which we denote by r_i, is given by the equation:

$$r_i = \frac{2k\|R_i\|}{1 - 2k\|R_i\| + \sqrt{1 - 4k\|R_i\|}}. \tag{3.11}$$

This value r_i satisfies $r_i \in (0, 1)$ for all $i \geq i_0$. Since $\|x_i'\| = \|e_i\| = 1$ and $\|S_i\| = \|R_i\|$ for all $i \in \mathbb{N}$, then from Theorem 2.1 (where we take $\delta = \|F\| = k$), the system of equations

$$Az_i = \lambda_i z_i, \qquad x_i'(z_i) = 1,$$

where $i \geq i_0$, has a unique solution (z_i, λ_i) in the set $\{z \in \mathscr{D}(D) : \|z - e_i\| \leq r_i\} \times \mathbb{C}$, where r_i is the smaller solution of equation (3.9) and is given by equation (3.11). Since $\|R_i\| \to 0$ as $i \to \infty$ then from equation (3.11), $r_i \to 0$ as $i \to \infty$. Hence $\|z_i - e_i\| \to 0$ as $i \to \infty$. Also from equation (2.3),

$$\lambda_i - a_{ii} = x_i'(F(z_i)). \tag{3.12}$$

But since Fe_i is in the closure of the set $\operatorname{span}\{e_j : j \neq i\}$ and $x_i'(e_j) = \delta_{ij}$, then $\lambda_i - a_{ii} = x_i'(F(z_i - e_i))$. Thus we have

$$|\lambda_i - a_{ii}| \leq \|x_i'\| \, \|F\| \, \|z_i - e_i\| \leq kr_i. \tag{3.13}$$

But since $r_i \to 0$ as $i \to \infty$, then $|\lambda_i - a_{ii}| \to 0$ as $i \to \infty$ and this completes the proof of the theorem.

Remark 3.3. If hypothesis (2) of Theorem 3.2 is replaced by

$$Q_i = \sum_j{}' |a_{ji}| = \sigma_i |a_{ii}|, \quad \text{where } \sigma_i \in [0, 1],$$

for all $i \in \mathbb{N}$, we get a similar conclusion to that given in the theorem where the sets $\mathscr{R}_i$ are replaced by $\mathscr{C}_i$, $i \in \mathbb{N}$.

4. Approximation of the eigenvalues of Mathieu's equation

In [1], we have shown that the eigenvalues of Mathieu's equation:

$$d^2y/d\theta^2 + (a - 2\cos 2\theta)y = 0$$

corresponding to the eigenfunctions $ce_{2n}(\theta, 1)$ are the eigenvalues of the infinite matrix operator $B = (b_{ij})$ defined in l_2 by:

$$b_{12} = b_{21} = \sqrt{2}, \qquad b_{23} = 1$$

and

$$b_{ij} = \begin{cases} 1 & \text{if } j = 1 \pm 1,\ i \geq 3, \\ 4(i-1)^2 & \text{if } j = i,\ i \geq 1, \\ 0 & \text{otherwise}. \end{cases}$$

Let $A = (a_{ij}) = B + 2I$. It is clear that A satisfies all the hypotheses of Theorem 3.2. Some calculations show that the operator $F = ((1 - \delta_{ij})a_{ij})$ has norm equal to 2. Using the same notations introduced in Theorem 3.2, the operator R_i (which is the inverse of the restriction of $(D - a_{ii}I)$ to the space $\mathscr{X}_i$ and $D = \operatorname{diag}(a_{11}, a_{22}, \dots))$ has a norm:

$$\|R_i\| = \frac{1}{4[(i-1)^2 - (i-2)^2]} = \frac{1}{4(2i-3)}$$

for all $i \geq 2$ and $\|R_1\| = 1/4$. For all $i \geq 3$, R_i satisfies equation (3.8). Hence from Theorem 3.2, the set of the eigenvalues of A is a discrete, countable set $\{\lambda_i : i \in \mathbb{N}\}$, where for all $i \geq 3$:

$$|\lambda_i - (2 + 4(i-1)^2)| \leq 2r_i \tag{4.1}$$

(see equation (3.13) and notice that $\|F\| = k = 2$) and r_i is given by

$$r_i = 1 \Big/ \left(2(i-1) + \sqrt{4(i-2)^2 - 1}\right) \tag{4.2}$$

(see equation (3.11)). Since the eigenvalues of B are $\mu_i = \lambda_i - 2$, $i \in \mathbb{N}$, then from equations (4.1) and (4.2), we have for all $i \geq 3$:

$$|\mu_i - 4(i-1)^2| \leq 2r_i$$

and $r_i \to 0$ as $i \to \infty$. For example,

for $i = 3$:

$$r_3 = 2 - \sqrt{3} < 0.268.$$

Hence $|\mu_3 - 16| < 0.536$.

For $i = 4$:

$$r_4 = 4 - \sqrt{15} < 0.1271.$$

For $i = 10$:

$$r_{10} = 16 - \sqrt{255} < 0.032.$$

Hence $|\mu_{10} - 324| < 0.064$.

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