ON $r$-SEPARATED SETS IN NORMED SPACES

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Abstract. The separation of a bounded set $A$ in a metric space $\delta(A)$ is defined as the supremum of the numbers $r > 0$ such that there exists a sequence $(x_n)$ in $A$ such that $d(x_n, x_m) > r$ for every $n \neq m$. We prove for every bounded set $A$ in a Banach space that $\delta(A) = \delta(\text{co}(A))$ where $\text{co}(A)$ denotes the convex hull of $A$. This yields a generalization of Darbo's fixed point theorem.

1. Introduction

In 1939 Kuratowski [10] introduced the measure of noncompactness $\alpha(A)$ of a bounded set $A$ in a metric space $X$. $\alpha(A)$ is called the Kuratowski measure of noncompactness, and is defined as the greatest lower bound of the numbers $r > 0$ such that $A$ can be decomposed into a finite union of sets of diameter smaller than $r$. The condition $\alpha(A) = 0$ therefore means that $A$ is precompact. Another measure of noncompactness, which in many cases seems to be more convenient, is called the ball-measure, $\beta(A)$, and is defined as the infimum of the real numbers $r > 0$ such that there is a finite cover of $A$ with balls of radii smaller than $r$.

These and other measures of noncompactness were used by Darbo [2], Massat [11], Sadovskii [12], and Banaś and Goebel [1] to obtain some fixed point theorems of nonlinear maps. In general, a measure of noncompactness on a complete metric space $X$ is a function $\gamma$ which maps every bounded set $B \subset X$ to a positive real number $\gamma(B)$ such that:

(a) $\gamma(B) = 0$ if and only if $\overline{B}$ is compact;
(b) if $B \subset C$ are bounded sets, then $\gamma(B) \leq \gamma(C)$.

Furthermore, if $B$ is a closed convex bounded subset of a Banach space $X$, an operator $T: B \rightarrow B$ is called $\gamma$-condensing if for all bounded sets $C \subset B$ we have $\gamma(T(C)) \leq \gamma(C)$ with equality if and only if $\gamma(C) = 0$.

The following theorem (Sadovskii [12], Massat [11]) illustrates the utility of these concepts.

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Theorem. Let $X$ be a Banach space and $B \subset X$ a closed convex bounded set. Let $T: B \to B$ be a continuous and $\gamma$-condensing operator, where $\gamma$ is a measure of noncompactness on $X$ such that

(c) $\gamma(A \cup B) = \gamma(A)$ for every finite set $B \subset X$;

(d) $\gamma(A) = \gamma(\text{co}(A))$ for every bounded set $A \subset X$, where $\text{co}(A)$ denotes the convex hull of $A$.

Then $T$ has a fixed point.

In Wells and Williams [13], another measure of noncompactness is defined: the separation of $A$, $\delta(A)$, is the supremum of the numbers $r > 0$ such that there exists a sequence $(x_n)$ in $A$ such that $d(x_n, x_m) > r$ for every $n \neq m$. They use it because this measure can distinguish between the unit balls of Banach spaces. The measure of the unit ball seems to be connected with the reflexivity of a Banach space (Kottman [9]). And the theorem of Elton and Odell [6] insures that $\delta(B) > 1$ for the unit ball $B$ of every infinite dimensional Banach space.

This same concept of separation of $A$ has been defined independently by Domínguez Benavides [3] and is denoted by $\mu(A)$. He observes [4] that in some Banach spaces as the $l_p$ spaces ($1 < p < +\infty$), $\delta(A) = \mu(A)$ is proportional to the $\beta$-measure. In other spaces, say $L^p[0, 1]$ ($p \neq 2$), this relation is not satisfied.

Observe that the above fixed point theorem would apply to this measure of noncompactness if we proved that

$$\delta(A) = \delta(\text{co}(A))$$

for every bounded subset $A$ of a Banach space. The purpose of this paper is to prove this. It is important to point out that, in a recent paper, Domínguez Benavides [5] proves that every $\alpha$-contraction is a $\delta$-contraction, so the fixed point theorem obtained in our paper generalizes the fixed point theorem of Darbo [2].

In §2 we give a probabilistic lemma (Corollary 2) that plays an essential role in the proof of the main theorem (Theorem 5). In that proof we also use an easier version of a deep theorem of D. H. Fremlin and M. Talagrand about random graphs, which we enunciate as Theorem 3.

2. Probabilistic lemmas

Lemma 1. Let $\mu, \nu$ be two probability measures on the space $[-1, 1]$ such that

$$\int_{-1}^{1} x \, d\mu(x) - \int_{-1}^{1} x \, d\nu(x) \geq s > \theta > 1,$$

then there exists a real number $t_0 \in [\theta - 1, 1]$ such that

$$\mu([-1, t_0]) + \nu([t_0 - \theta, 1]) \leq (2 - s)/(2 - \theta).$$
Proof. Observe that an elementary calculation leads us to
\[\int_{-1}^{1} (1 + x) \, d\mu(x) = \int_{0}^{2} \mu[-1 + t, 1] \, dt = \int_{-1}^{1} \mu[t, 1] \, dt.\]

It follows that
\[\int_{-1}^{1} x \, d\mu(x) = 1 - \int_{-1}^{1} \mu[-1, t] \, dt\]
\[\int_{-1}^{1} x \, d\nu(x) = \int_{-1}^{1} \nu[t, 1] \, dt - 1.\]

Therefore
\[\int_{-1}^{1} x \, d\mu(x) - \int_{-1}^{1} x \, d\nu(x) = 2 - \int_{-1}^{1} \mu[-1, t] - \int_{-1}^{1} \nu[t, 1] \, dt\]
\[= 2 - \int_{-1}^{1} \mu[-1, t] - \int_{-1}^{1} \nu[t - \theta, 1] \, dt\]
\[\leq 2 - \int_{-1}^{1+\theta} (\mu[-1, t] + \nu[t - \theta, 1]) \, dt.\]

Define \(\psi(t) = (\mu[-1, t] + \nu[t - \theta, 1]).\) By the hypothesis we know that
\[2 - s \geq \int_{-1+\theta}^{1} \psi(t) \, dt.\]

Then \(\psi(t) > (2 - s)/(2 - \theta)\) for every \(t \in [-1 + \theta, 1]\) would lead to a contradiction. It follows that there exists \(t_0 \in [-1 + \theta, 1]\) such that \(\psi(t_0) \leq (2 - s)/(2 - \theta).\)

We shall use the following corollary in the proof of the main theorem.

Corollary 2. Let \((\Omega_1, \mathbb{P}_1), (\Omega_2, \mathbb{P}_2)\) be two probability spaces and \(X_i : \Omega_i \to E\) two random variables with values in the same Banach space \(E.\) Assume that
\[\|X_i\|_{\infty} \leq 1 \quad \text{and} \quad \|E(X_1) - E(X_2)\| \geq s > \theta > 1.\]

Then there exist two measurable subsets \(A \subset \Omega_1\) and \(B \subset \Omega_2\) such that

1. \(\mathbb{P}_1(A) + \mathbb{P}_2(B) \leq (2 - s)/(2 - \theta) < 1.\)
2. \(\omega_1 \notin A \text{ and } \omega_2 \notin B \text{ implies } \|X_1(\omega_1) - X_2(\omega_2)\| > \theta.\)

Proof. There is no loss of generality in assuming that the Banach space \(E\) is real. Let \(x^* \in E^*\) be a vector of the dual space such that \(\|x^*\| = 1\) and
\[\|E(X_1) - E(X_2)\| = E(x^* \circ X_1) - E(x^* \circ X_2).\]

Let \(\mu\) and \(\nu\) be the image of the measures \(\mathbb{P}_1\) and \(\mathbb{P}_2\) under the mappings \(x^* \circ X_1\) and \(x^* \circ X_2.\) Since \(\|X_i\|_{\infty} \leq 1, \mu\) and \(\nu\) are probability measures on \([-1, 1].\) Observe that
\[\int_{-1}^{1} x \, d\mu(x) - \int_{-1}^{1} x \, d\nu(x) = E(x^* \circ X_1) - E(x^* \circ X_2) \geq s > \theta > 1.\]
Applying the lemma we find $t_0 \in [\theta - 1, 1]$ such that

$$
\mu[-1, t_0] + \nu[t_0 - \theta, 1] \leq (2 - s)/(2 - \theta) < 1.
$$

Define $A = (x^* \circ X_1)^{-1}[-1, t_0]$ and $B = (x^* \circ X_2)^{-1}[t_0 - \theta, 1]$. Thus $A \subset \Omega_1$ and $B \subset \Omega_2$ are measurable sets that satisfy (1).

If $\omega_1 \notin A$ and $\omega_2 \notin B$, then $x^*(X_1(\omega_1)) \geq t_0$ and $x^*(X_2(\omega_2)) < t_0 - \theta$.

It follows that

$$
\|X_1(\omega_1) - X_2(\omega_2)\| \geq x^*(X_1(\omega_1) - X_2(\omega_2)) > \theta
$$

and (2) is also satisfied.

We shall use the following consequence of a theorem of D. H. Fremlin and M. Talagrand [7]:

**Theorem 3.** Let $(\Omega_n, \mathbb{P}_n)$ be a sequence of probability spaces. For every $m < n$, let $B_{n,m} \subset \Omega_n$ be a measurable set such that $\mathbb{P}_n(B_{n,m}) < \alpha < 1$. Then there exists an infinite set $J \subset \mathbb{N}$ such that, for every $n \in J$,

$$
\mathbb{P}_n \left( \bigcup_{m < n, m \in J} B_{n,m} \right) < 1.
$$

**Proof.** We apply the theorems 6C and 6D [7] to the probability space $\Omega = \prod_{j=1}^{\infty} \Omega_j$ and the measurable sets

$$
E_{n,m} = \Omega \setminus \left( B_{n,m} \times \prod_{j \neq n} \Omega_j \right)
$$

in order to obtain the theorem.

3. **THEOREM ABOUT THE MEASURE OF NONCOMPACTNESS**

Let us recall that if $A \subset X$ is a bounded subset in a metric space $X$, the separation of $A$, $\delta(A)$ is defined as the supremum of the real numbers $r > 0$ such that there exists a sequence $(x_n)$ in $A$ verifying $\|x_n - x_m\| > r$ for every two distinct elements $x_n, x_m$ of the sequence.

We begin making a reduction in the possible counterexample to $\delta(A) = \delta(\text{co}(A))$.

**Proposition 4.** Let $A$ be a bounded subset of the normed space $X$ such that $\delta(A) < \delta(\text{co}(A))$. Then, for every $s$ verifying $\delta(A) < s < \delta(\text{co}(A))$, there exists a set $B$ contained in the ball of center 0 and radius $s$ and such that

$$
\delta(B) \leq \delta(A) < \delta(\text{co}(A)) = \delta(\text{co}(B)).
$$

**Proof.** Take a maximal set $\{a_k\}_{k=1}^N$ of points belonging to $A$ such that $i \neq j$ implies $\|a_i - a_j\| \geq s$. The number $N$ of elements is finite since $\delta(A) < s$. 

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It is clear that $A$ is contained in the union of the balls $B(a_k, s)$ of center $a_k$ and radius $s$:

$$A \subset \bigcup_{k=1}^{N} B(a_k, s).$$

Every point $x \in \text{co}(A)$ can be written in the form $x = \sum_{k=1}^{N} \alpha_k x_k$, where $\alpha_k \geq 0$, $\sum \alpha_k = 1$ and $x_k \in \text{co}(A \cap B(a_k, s))$. Now, given $\varepsilon > 0$, let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in $\text{co}(A)$ such that $n \neq m$ implies $\|z_n - z_m\| > \delta(\text{co}(A)) - \varepsilon$.

Write every $z_n$ in the form:

$$z_n = \sum_{k=1}^{N} \alpha_k^n y_k^n,$$

where $\alpha_k^n \geq 0$, $\sum_{k=1}^{N} \alpha_k^n = 1$, and $y_k^n \in \text{co}(A \cap B(a_k, s))$. Choose an infinite subset $I \subset \mathbb{N}$ such that if $n, m \in I$, then for all $k \leq N$,

$$|\alpha_k^n - \alpha_k^m| < \varepsilon / (N \cdot \sup_{a \in A} \|a\|).$$

Now let $n, m \in I$ and $n \neq m$. Then there exists a natural number $k$ such that $\|y_k^n - y_k^m\| > \delta(\text{co}(A)) - 2\varepsilon$, because otherwise

$$\|z_n - z_m\| \leq \left\| \sum_{k=1}^{N} \alpha_k^n (y_k^n - y_k^m) + (\alpha_k^n - \alpha_k^m) y_k^m \right\|$$

$$\leq (\delta(\text{co}(A)) - 2\varepsilon) \cdot 1 + \sup_{a \in A} \|a\| \sum_{k=1}^{N} |\alpha_k^n - \alpha_k^m|$$

$$< \delta(\text{co}(A)) - \varepsilon,$$

which contradicts the hypothesis.

Applying the Ramsey theorem, T. Jech [8], we obtain an infinite set $J$ contained in $I$ and an index $k$ such that $n, m \in J$, $n \neq m$ imply $\|y_k^n - y_k^m\| > \delta(\text{co}(A)) - 2\varepsilon$. Therefore, for every $\varepsilon > 0$, there exists an index $k$, $1 \leq k \leq N$, such that $\delta(\text{co}(A \cap B(a_k, s))) \geq \delta(\text{co}(A)) - 2\varepsilon$. Hence there exists an index $k$ such that

$$\delta(\text{co}(A \cap B(a_k, s))) \geq \delta(\text{co}(A)).$$

Furthermore we observe that $\delta(A \cap B(a_k, s)) \leq \delta(A)$. We can translate the set $A \cap B(a_k, s)$ and obtain the set $B = -a_k + A \cap B(a_k, s)$, which is contained in the ball of center 0 and radius $s$ and satisfies

$$\delta(B) \leq \delta(A) < \delta(\text{co}(A)) \leq \delta(\text{co}(B)).$$

**Theorem 5.** Let $A \subset X$ be a bounded subset of a normed space $X$. Then $\delta(A) = \delta(\text{co}(A))$.

**Proof.** Suppose that the theorem is false. Then Proposition 4 implies that there exists a subset $A$ of the unit ball of a Banach space $E$, such that

$$\delta(A) < 1 < s < \delta(\text{co}(A)).$$
We can take a sequence \((x_n)\) in \(\text{co}(A)\) such that for every \(n \neq m\), \(\|x_n - x_m\| \geq s\). As \(x_n \in \text{co}(A)\), there exists a finite subset \(\Omega_n \subset A\) and, for every \(e \in \Omega_n\), a real number \(\alpha_e > 0\) such that

\[
x_n = \sum_{e \in \Omega_n} \alpha_e e \quad \text{and} \quad \sum_{e \in \Omega_n} \alpha_e = 1.
\]

Let \(\mathbb{P}_n\) be the probability defined on \(\Omega_n\) by

\[
\mathbb{P}_n(B) = \sum_{e \in B} \alpha_e,
\]

for every \(B \subset \Omega_n\).

For every \(n\), let \(X_n : \Omega_n \to E\) be the random variable, defined as the identity in \(\Omega_n\). It is clear that if we choose \(\theta\) such that \(1 < \theta < s\), we have, for every \(n \neq m\),

\[
\|\mathbb{E}(X_n) - \mathbb{E}(X_m)\| = \|x_n - x_m\| > s > \theta > 1.
\]

We are now in a position to apply our probabilistic lemma and find subsets \(B_{n,m} \subset \Omega_n\) and \(B_{m,n} \subset \Omega_m\) verifying:

(a) \(\mathbb{P}_n(B_{n,m}) + \mathbb{P}_m(B_{m,n}) \leq (2-s)/(2-\theta) < 1\).

(b) \(e_n \notin B_{n,m}\) and \(e_m \notin B_{m,n}\) implies \(\|e_n - e_m\| \geq \theta\).

For every natural number \(m_0\), \((B_{m_0,n}/n > m_0)\) is a sequence of subsets of the finite set \(\Omega_{m_0}\). It follows that we can find an infinite set \(J \subset \mathbb{N}\) such that \(B_{m_0,n}\) is independent of \(n \in J\). By a diagonal argument we can obtain an infinite set \(J \subset \mathbb{N}\) such that if \(n > m\), \(m, n \in J\), \(B_{m,n}\) is independent of \(n\). We call it \(B_m\). Now by exchanging the sequence \((x_n)_{n \in \mathbb{N}}\) for \((x_n)_{n \in J}\), we can also assume that the original sequence satisfies these conditions.

Let \(\delta = (2-s)/(2-\theta) < 1\), and choose \(0 < \varepsilon < (1-\delta)/2\). We can assume that \(\lim \mathbb{P}_n(B_n)\) exists, call it \(l\). Thus \(l = \lim \mathbb{P}_n(B_n) \leq 1\); even more, we can assume \(|\mathbb{P}_n(B_n) - l| < \varepsilon\) for every natural number \(n\).

Now, for every pair of natural numbers \(n > m\)

\[
\mathbb{P}_n(B_{n,m}) + \mathbb{P}_m(B_m) \leq (2-s)/(2-\theta) = \delta < 1.
\]

Hence \(l \leq \delta\) and \(l - \varepsilon < \mathbb{P}_m(B_m) \leq \delta\).

Therefore we can consider the spaces \(\Omega'_n = \Omega_n \setminus B_n\) endowed with the measures \(\mathbb{P}'_n = (1 - \mathbb{P}_n(B_n))^{-1} \mathbb{P}_n\). This measure is well defined because \(\mathbb{P}_n(B_n) \leq \delta < 1\). Now, for every \(m < n\),

\[
\mathbb{P}'_n(B_{n,m} \setminus B_n) = \frac{\mathbb{P}_n(B_{n,m} \setminus B_n)}{1 - \mathbb{P}_n(B_n)} \leq \frac{\mathbb{P}_n(B_{n,m})}{1 - \mathbb{P}_n(B_n)} \leq \frac{\delta - \mathbb{P}_m(B_m)}{1 - \mathbb{P}_n(B_n)} \leq \frac{1 - \mathbb{P}_n(B_n) + (\mathbb{P}_n(B_n) - \mathbb{P}_m(B_m)) - (1 - \delta)}{1 - \mathbb{P}_n(B_n)} \leq 1 - \frac{1 - \delta - 2\varepsilon}{1 - \mathbb{P}_n(B_n)} \leq 1 - \frac{1 - \delta - 2\varepsilon}{1 - l + \varepsilon} = 1 - \alpha < 1.
\]
Hence the conditions in the hypothesis of the D. H. Fremlin and M. Talagrand theorem are satisfied. So, we obtain an infinite set $J \subset \mathbb{N}$ such that, for every $n \in J$, 

$$\mathbb{P}_n \left( \bigcup_{m \in J, m < n} (B_{n,m} \setminus B_n) \right) < 1.$$ 

Now, if we put $B_n = B_{n,n}$, it follows that 

$$\mathbb{P}_n \left( \bigcup_{m \in J} B_{n,m} \right) < 1.$$ 

Thus there exists, for every $n \in J$, $e_n \in \Omega_n$ such that 

$$e_n \notin \bigcup_{m \in J} B_{n,m}.$$ 

Now if $n > m$ and $n, m \in J$, we have $e_n \notin B_{n,m}$ and $e_m \notin B_{m,m}$. Therefore $e_m \notin B_{m,n}$. Condition (b) insures that $\|e_n - e_m\| \geq \theta$. We have thus found a sequence of points $(e_n)_{n \in J}$ in $A$, such that for every $n \neq m$, $n, m \in J$, $\|e_n - e_m\| \geq \theta > 1$ which contradicts $\delta(A) < 1$.

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**References**


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