ON $r$-SEPARATED SETS IN NORMED SPACES

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Abstract. The separation of a bounded set $A$ in a metric space $\delta(A)$ is defined as the supremum of the numbers $r > 0$ such that there exists a sequence $(x_n)$ in $A$ such that $d(x_n, x_m) > r$ for every $n \neq m$. We prove for every bounded set $A$ in a Banach space that $\delta(A) = \delta(\co(A))$ where $\co(A)$ denotes the convex hull of $A$. This yields a generalization of Darbo's fixed point theorem.

1. Introduction

In 1939 Kuratowski [10] introduced the measure of noncompactness $\alpha(A)$ of a bounded set $A$ in a metric space $X$. $\alpha(A)$ is called the Kuratowski measure of noncompactness, and is defined as the greatest lower bound of the numbers $r > 0$ such that $A$ can be decomposed into a finite union of sets of diameter smaller than $r$. The condition $\alpha(A) = 0$ therefore means that $A$ is precompact. Another measure of noncompactness, which in many cases seems to be more convenient, is called the ball-measure, $\beta(A)$, and is defined as the infimum of the real numbers $r > 0$ such that there is a finite cover of $A$ with balls of radii smaller than $r$.

These and other measures of noncompactness were used by Darbo [2], Massat [11], Sadovskii [12], and Banaś and Goebel [1] to obtain some fixed point theorems of nonlinear maps. In general, a measure of noncompactness on a complete metric space $X$ is a function $\gamma$ which maps every bounded set $B \subset X$ to a positive real number $\gamma(B)$ such that:

(a) $\gamma(B) = 0$ if and only if $\overline{B}$ is compact;
(b) if $B \subset C$ are bounded sets, then $\gamma(B) \leq \gamma(C)$.

Furthermore, if $B$ is a closed convex bounded subset of a Banach space $X$, an operator $T: B \to B$ is called $\gamma$-condensing if for all bounded sets $C \subset B$ we have $\gamma(T(C)) \leq \gamma(C)$ with equality if and only if $\gamma(C) = 0$.

The following theorem (Sadovskii [12], Massat [11]) illustrates the utility of these concepts.

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Theorem. Let $X$ be a Banach space and $B \subset X$ a closed convex bounded set. Let $T : B \to B$ be a continuous and $\gamma$-condensing operator, where $\gamma$ is a measure of noncompactness on $X$ such that

(c) $\gamma(A \cup B) = \gamma(A)$ for every finite set $B \subset X$;
(d) $\gamma(A) = \gamma(\text{co}(A))$ for every bounded set $A \subset X$, where $\text{co}(A)$ denotes the convex hull of $A$.

Then $T$ has a fixed point.

In Wells and Williams [13], another measure of noncompactness is defined: the separation of $A$, $\delta(A)$, is the supremum of the numbers $r > 0$ such that there exists a sequence $(x_n)$ in $A$ such that $d(x_n, x_m) > r$ for every $n \neq m$. They use it because this measure can distinguish between the unit balls of Banach spaces. The measure of the unit ball seems to be connected with the reflexivity of a Banach space (Kottman [9]). And the theorem of Elton and Odell [6] insures that $\delta(B) > 1$ for the unit ball $B$ of every infinite dimensional Banach space.

This same concept of separation of $A$ has been defined independently by Domínguez Benavides [3] and is denoted by $\mu(A)$. He observes [4] that in some Banach spaces as the $l_p$ spaces ($1 < p < +\infty$), $\delta(A) = \mu(A)$ is proportional to the $\beta$-measure. In other spaces, say $L^p[0, 1]$ ($p \neq 2$), this relation is not satisfied.

Observe that the above fixed point theorem would apply to this measure of noncompactness if we proved that

$$\delta(A) = \delta(\text{co}(A))$$

for every bounded subset $A$ of a Banach space. The purpose of this paper is to prove this. It is important to point out that, in a recent paper, Domínguez Benavides [5] proves that every $\alpha$-contraction is a $\delta$-contraction, so the fixed point theorem obtained in our paper generalizes the fixed point theorem of Darbo [2].

In §2 we give a probabilistic lemma (Corollary 2) that plays an essential role in the proof of the main theorem (Theorem 5). In that proof we also use an easier version of a deep theorem of D. H. Fremlin and M. Talagrand about random graphs, which we enunciate as Theorem 3.

2. Probabilistic lemmas

Lemma 1. Let $\mu, \nu$ be two probability measures on the space $[-1, 1]$ such that

$$\int_{-1}^{1} x \, d\mu(x) - \int_{-1}^{1} x \, d\nu(x) \geq s > \theta > 1,$$

then there exists a real number $t_0 \in [\theta - 1, 1]$ such that

$$\mu[-1, t_0] + \nu[t_0 - \theta, 1] \leq (2 - s)/(2 - \theta).$$
Proof. Observe that an elementary calculation leads us to
\[
\int_{-1}^{1} (1 + x) \, d\mu(x) = \int_{0}^{2} \mu[-1 + t, 1] \, dt = \int_{-1}^{1} \mu[t, 1] \, dt.
\]
It follows that
\[
\int_{-1}^{1} x \, d\mu(x) = 1 - \int_{-1}^{1} \mu[-1, t] \, dt
\]
\[
\int_{-1}^{1} x \, d\nu(x) = \int_{-1}^{1} \nu[t, 1] \, dt - 1.
\]
Therefore
\[
\int_{-1}^{1} x \, d\mu(x) - \int_{-1}^{1} x \, d\nu(x) = 2 - \int_{-1}^{1} \mu[-1, t] - \int_{-1}^{1} \nu[t, 1] \, dt
\]
\[
= 2 - \int_{-1}^{1} \mu[-1, t] - \int_{-1}^{1} \nu[t - \theta, 1] \, dt
\]
\[
\leq 2 - \int_{-1}^{1} (\mu[-1, t] + \nu[t - \theta, 1]) \, dt.
\]
Define \(\psi(t) = (\mu[-1, t] + \nu[t - \theta, 1])\). By the hypothesis we know that
\[
2 - s \geq \int_{-1}^{1} \psi(t) \, dt.
\]
Then \(\psi(t) > (2 - s)/(2 - \theta)\) for every \(t \in [-1 + \theta, 1]\) would lead to a contradiction. It follows that there exists \(t_0 \in [-1 + \theta, 1]\) such that \(\psi(t_0) \leq (2 - s)/(2 - \theta)\).

We shall use the following corollary in the proof of the main theorem.

Corollary 2. Let \((\Omega_1, \mathbb{P}_1), (\Omega_2, \mathbb{P}_2)\) be two probability spaces and \(X_i : \Omega_i \rightarrow E\) two random variables with values in the same Banach space \(E\). Assume that
\[
\|X_1\|_{\infty} \leq 1 \quad \text{and} \quad \|E(X_1) - E(X_2)\| \geq s > \theta > 1.
\]
Then there exist two measurable subsets \(A \subset \Omega_1\) and \(B \subset \Omega_2\) such that
\[
P_1(A) + P_2(B) \leq (2 - s)/(2 - \theta) < 1.
\]
(2) \(\omega_1 \notin A\) and \(\omega_2 \notin B\) implies \(\|X_1(\omega_1) - X_2(\omega_2)\| > \theta\).

Proof. There is no loss of generality in assuming that the Banach space \(E\) is real. Let \(x^* \in E^*\) be a vector of the dual space such that \(\|x^*\| = 1\) and
\[
\|E(X_1) - E(X_2)\| = E(x^* \circ X_1) - E(x^* \circ X_2).
\]
Let \(\mu\) and \(\nu\) be the image of the measures \(P_1\) and \(P_2\) under the mappings \(x^* \circ X_1\) and \(x^* \circ X_2\). Since \(\|X_i\|_{\infty} \leq 1\), \(\mu\) and \(\nu\) are probability measures on \([-1, 1]\). Observe that
\[
\int_{-1}^{1} x \, d\mu(x) - \int_{-1}^{1} x \, d\nu(x) = E(x^* \circ X_1) - E(x^* \circ X_2) \geq s > \theta > 1.
\]
Applying the lemma we find \( t_0 \in [\theta - 1, 1] \) such that
\[
\mu[-1, t_0] + \nu[t_0 - \theta, 1] \leq (2-s)/(2-\theta) < 1.
\]
Define \( A = (x^* \circ X_1)^{-1}[-1, t_0] \) and \( B = (x^* \circ X_2)^{-1}[t_0 - \theta, 1] \). Thus \( A \subset \Omega_1 \) and \( B \subset \Omega_2 \) are measurable sets that satisfy (1).

If \( \omega_1 \notin A \) and \( \omega_2 \notin B \), then \( x^*(X_1(\omega_1)) \geq t_0 \) and \( x^*(X_2(\omega_2)) < t_0 - \theta \).

It follows that
\[
\|X_1(\omega_1) - X_2(\omega_2)\| \geq x^*(X_1(\omega_1) - X_2(\omega_2)) > \theta
\]
and (2) is also satisfied.

We shall use the following consequence of a theorem of D. H. Fremlin and M. Talagrand [7]:

**Theorem 3.** Let \((\Omega_n, \mathbb{P}_n)\) be a sequence of probability spaces. For every \( m < n \), let \( B_{n,m} \subset \Omega_n \) be a measurable set such that \( \mathbb{P}_n(B_{n,m}) \leq \alpha < 1 \). Then there exists an infinite set \( J \subset \mathbb{N} \) such that, for every \( n \in J \),
\[
\mathbb{P}_n\left( \bigcup_{m<n} B_{n,m} \right) < 1.
\]

**Proof.** We apply the theorems 6C and 6D [7] to the probability space \( \Omega = \prod_{j=1}^\infty \Omega_j \) and the measurable sets
\[
E_{n,m} = \Omega \setminus \left( B_{n,m} \times \prod_{j \neq n} \Omega_j \right)
\]
in order to obtain the theorem.

3. **Theorem about the measure of noncompactness**

Let us recall that if \( A \subset X \) is a bounded subset in a metric space \( X \), the separation of \( A \), \( \delta(A) \) is defined as the supremum of the real numbers \( r > 0 \) such that there exists a sequence \( (x_n) \) in \( A \) verifying \( \|x_n - x_m\| > r \) for every two distinct elements \( x_n, x_m \) of the sequence.

We begin making a reduction in the possible counterexample to \( \delta(A) = \delta(\text{co}(A)) \).

**Proposition 4.** Let \( A \) be a bounded subset of the normed space \( X \) such that \( \delta(A) < \delta(\text{co}(A)) \). Then, for every \( s \) verifying \( \delta(A) < s < \delta(\text{co}(A)) \), there exists a set \( B \) contained in the ball of center 0 and radius \( s \) and such that
\[
\delta(B) \leq \delta(A) < \delta(\text{co}(A)) = \delta(\text{co}(B)).
\]

**Proof.** Take a maximal set \( \{a_k\}_{k=1}^N \) of points belonging to \( A \) such that \( i \neq j \) implies \( \|a_i - a_j\| \geq s \). The number \( N \) of elements is finite since \( \delta(A) < s \).
It is clear that $A$ is contained in the union of the balls $B(a_k, s)$ of center $a_k$ and radius $s$:

$$A \subset \bigcup_{k=1}^{N} B(a_k, s).$$

Every point $x \in \text{co}(A)$ can be written in the form $x = \sum_{k=1}^{N} \alpha_k x_k$, where $\alpha_k \geq 0$, $\sum \alpha_k = 1$ and $x_k \in \text{co}(A \cap B(a_k, s))$. Now, given $\varepsilon > 0$, let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in $\text{co}(A)$ such that $n \neq m$ implies $\|z_n - z_m\| > \delta(\text{co}(A)) - \varepsilon$.

Write every $z_n$ in the form:

$$z_n = \sum_{k=1}^{N} \alpha^n_k y^n_k,$$

where $\alpha^n_k \geq 0$, $\sum_{k=1}^{N} \alpha^n_k = 1$, and $y^n_k \in \text{co}(A \cap B(a_k, s))$. Choose an infinite subset $I \subset \mathbb{N}$ such that if $n, m \in I$, then for all $k \leq N$,

$$|\alpha^n_k - \alpha^m_k| < \varepsilon/(N \cdot \sup_{a \in A} \|a\|).$$

Now let $n, m \in I$ and $n \neq m$. Then there exists a natural number $k$ such that $\|y^n_k - y^m_k\| > \delta(\text{co}(A)) - 2\varepsilon$, because otherwise

$$\|z_n - z_m\| \leq \left\| \sum_{k=1}^{N} \alpha^n_k (y^n_k - y^m_k) + (\alpha^n_k - \alpha^m_k) y^m_k \right\|$$

$$\leq (\delta(\text{co}(A)) - 2\varepsilon) \cdot 1 + \sup_{a \in A} \|a\| \sum_{k=1}^{N} |\alpha^n_k - \alpha^m_k|$$

$$< \delta(\text{co}(A)) - \varepsilon,$$

which contradicts the hypothesis.

Applying the Ramsey theorem, T. Jech [8], we obtain an infinite set $J$ contained in $I$ and an index $k$ such that $n, m \in J$, $n \neq m$ imply $\|y^n_k - y^m_k\| > \delta(\text{co}(A)) - 2\varepsilon$. Therefore, for every $\varepsilon > 0$, there exists an index $k$, $1 \leq k \leq N$, such that $\delta(\text{co}(A \cap B(a_k, s))) \geq \delta(\text{co}(A)) - 2\varepsilon$. Hence there exists an index $k$ such that

$$\delta(\text{co}(A \cap B(a_k, s))) \geq \delta(\text{co}(A)).$$

Furthermore we observe that $\delta(A \cap B(a_k, s)) \leq \delta(A)$. We can translate the set $A \cap B(a_k, s)$ and obtain the set $B = -a_k + A \cap B(a_k, s)$, which is contained in the ball of center 0 and radius $s$ and satisfies

$$\delta(B) \leq \delta(A) < \delta(\text{co}(A)) \leq \delta(\text{co}(B)).$$

**Theorem 5.** Let $A \subset X$ be a bounded subset of a normed space $X$. Then $\delta(A) = \delta(\text{co}(A))$.

**Proof.** Suppose that the theorem is false. Then Proposition 4 implies that there exists a subset $A$ of the unit ball of a Banach space $E$, such that

$$\delta(A) < 1 < s < \delta(\text{co}(A)).$$
We can take a sequence \( (x_n) \) in \( c_0(A) \) such that for every \( n \neq m \), \( \|x_n - x_m\| \geq s \). As \( x_n \in c_0(A) \), there exists a finite subset \( \Omega_n \subseteq A \) and, for every \( e \in \Omega_n \), a real number \( \alpha_e > 0 \) such that

\[
x_n = \sum_{e \in \Omega_n} \alpha_e e \quad \text{and} \quad \sum_{e \in \Omega_n} \alpha_e = 1.
\]

Let \( P_n \) be the probability defined on \( \Omega_n \) by

\[
P_n(B) = \sum_{e \in B} \alpha_e,
\]

for every \( B \subseteq \Omega_n \).

For every \( n \), let \( X_n : \Omega_n \to E \) be the random variable, defined as the identity in \( \Omega_n \). It is clear that if we choose \( \theta \) such that \( 1 < \theta < s \), we have, for every \( n \neq m \),

\[
\|E(X_n) - E(X_m)\| = \|x_n - x_m\| > s > \theta > 1.
\]

We are now in a position to apply our probabilistic lemma and find subsets \( B_{n,m} \subseteq \Omega_n \) and \( B_{m,n} \subseteq \Omega_m \) verifying:

(a) \( P_n(B_{n,m}) + P_m(B_{m,n}) \leq (2 - s)/(2 - \theta) < 1 \).

(b) \( e_n \notin B_{n,m} \) and \( e_m \notin B_{m,n} \) implies \( \|e_n - e_m\| \geq \theta \).

For every natural number \( m_0 \), \((B_{m_0,n}, n > m_0)\) is a sequence of subsets of the finite set \( \Omega_{m_0} \). It follows that we can find an infinite set \( J \subseteq N \) such that \( B_{m_0,n} \) is independent of \( n \in J \). By a diagonal argument we can obtain an infinite set \( J \subseteq N \) such that if \( n > m, m, n \in J \), \( B_{m,n} \) is independent of \( n \). We call it \( B_m \). Now by exchanging the sequence \( (x_n)_{n \in N} \) for \( (x_n)_{n \in J} \), we can also assume that the original sequence satisfies these conditions.

Let \( \delta = (2 - s)/(2 - \theta) < 1 \), and choose \( 0 < \epsilon < (1 - \delta)/2 \). We can assume that \( \lim P_n(B_n) \) exists, call it \( l \). Thus \( l = \lim P_n(B_n) \leq 1 \); even more, we can assume \( |P_n(B_n) - l| < \epsilon \) for every natural number \( n \).

Now, for every pair of natural numbers \( n > m \)

\[
P_n(B_{n,m}) + P_m(B_m) \leq (2 - s)/(2 - \theta) = \delta < 1.
\]

Hence \( l \leq \delta \) and \( l - \epsilon < P_m(B_m) \leq \delta \).

Therefore we can consider the spaces \( \Omega'_n = \Omega_n \setminus B_n \) endowed with the measures \( P'_n = (1 - P_n(B_n))^{-1} P_n \). This measure is well defined because \( P_n(B_n) \leq \delta < 1 \). Now, for every \( m < n \),

\[
P'_n(B_{n,m} \setminus B_n) = \frac{P_n(B_{n,m} \setminus B_n)}{1 - P_n(B_n)} \leq \frac{P_n(B_{n,m})}{1 - P_n(B_n)} \leq \frac{\delta - P_m(B_m)}{1 - P_n(B_n)} \leq \frac{1 - P_n(B_n) + (P_n(B_n) - P_m(B_m)) - (1 - \delta)}{1 - P_n(B_n)} \leq 1 - \delta - 2\epsilon \frac{1}{1 - l + \epsilon} = 1 - \alpha < 1.
\]
Hence the conditions in the hypothesis of the D. H. Fremlin and M. Talagrand theorem are satisfied. So, we obtain an infinite set \( J \subseteq \mathbb{N} \) such that, for every \( n \in J \),

\[
\mathbb{P}^n_\mathcal{F} \left( \bigcup_{m \leq n, m \in J} (B_{n,m} \setminus B_n) \right) < 1.
\]

Now, if we put \( B_n = B_{n,n} \), it follows that

\[
\mathbb{P}^n \left( \bigcup_{m \leq n, m \in J} B_{n,m} \right) < 1.
\]

Thus there exists, for every \( n \in J \), \( e_n \in \Omega_n \) such that

\[
e_n \notin \bigcup_{m \leq n, m \in J} B_{n,m}.
\]

Now if \( n > m \) and \( n, m \in J \), we have \( e_n \notin B_{n,m} \) and \( e_m \notin B_{m,m} \). Therefore \( e_m \notin B_{m,n} \). Condition (b) insures that \( \|e_n - e_m\| \geq \theta \). We have thus found a sequence of points \( (e_n)_{n \in J} \) in \( A \), such that for every \( n \neq m, n, m \in J \), \( \|e_n - e_m\| \geq \theta > 1 \) which contradicts \( \delta(A) < 1 \).

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