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# PRODUCTS OF PERFECTLY MEAGRE SETS

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ABSTRACT. We show that there exists a perfect set $D \subseteq 2^\omega \times 2^\omega$ such that for every Luzin set in $D$ both projections of it are perfectly meagre. It follows (under CH) that the product of two perfectly meagre sets need not be perfectly meagre (or even have the Baire property in the restricted sense). This provides an answer to a 55-year-old question of Marczewski.

A subset $L$ of a completely separable metric space is a (generalized) Luzin set iff $L \cap K$ is countable (less than the continuum) for every meagre set $K$ and $|L| > \omega$ $(|L| = c)$.

A subset $X$ of a completely separable metric space is perfectly meagre iff, for every perfect set $D$, $X \cap D$ is meagre in $D$. $X$ has the Baire property in the restricted sense if, for every perfect set $D$, $X \cap D$ has the Baire property (symmetric difference of an open set and a first-category set) relative to $D$.

$$\mathrm{non}(\mathbb{K}) = \{|X| : X \subseteq \mathbb{R} \text{ and } X \text{ is not meagre}\}.$$

In 1934, Sierpinski [Si2] showed (under CH) that the product of a set with the Baire property in the restricted sense and a complete metric space need not have the Baire property in the restricted sense. Sierpinski used the so-called function of Luzin [L] in his construction.

In 1935, E. Marczewski (Szpilrajn) asked a question: Is the product of two perfectly meagre sets perfectly meagre? (See [Mi] or [S].)

If there exists a Luzin set, then the answer to this question is "No."

**Theorem.** *There exists a perfect set $D \subseteq 2^\omega \times 2^\omega$ such that, for every perfect set $E \subseteq 2^\omega$, there exists $G \subseteq E$ dense $G_\delta$-set in $E$ such that $((G \times 2^\omega) \cap (2^\omega \times G)) \cap D$ is nowhere dense in $D$.*

**Corollary 1.** *Let $D$ be a perfect set as above.*

(1) *If $L \subseteq D$ is a Luzin set in $D$, or*
(2) *if $L \subseteq D$ is a generalized Luzin set in $D$ and* $\mathrm{non}(\mathbb{K}) = c$, *then $\pi_1(L)$ and $\pi_2(L)$ are perfectly meagre.*

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*Proof*. Let $E$ and $G \subseteq E$ have the properties from the thesis of the theorem. If $L$ is a Luzin set in $D$, then $|L \cap (G \times 2^\omega)| \leq \omega$. So $|(\pi_1(L) \cap G)| \leq \omega$. Thus $G \setminus \pi_1(L)$ is residual in $E$. The proof for the generalized Luzin set is similar.

**Corollary 2.** (1) *If there exists a Luzin set, or*

(2) *if there exists a generalized Luzin set and* $\operatorname{non}(\mathbb{K}) = c$*, then there exist* $X, Y \subseteq 2^\omega$ *perfectly meagre such that* $X \times Y$ *is not perfectly meagre.*

*Proof*. We can assume that $L$ is a Luzin set in $D$. Let $X = \pi_1(L)$ and $Y = \pi_2(L)$. Then $L \subseteq X \times Y$, so $X \times Y$ is second-category in $D$.

*Proof of Theorem*. Let $\{A_n : n \in \omega\}$ be a family of infinite pairwise disjoint subsets of $\omega$. We will define $D \subseteq 2^\omega \times 2^\omega$ such that $(x, y) \notin D \Leftrightarrow \exists_n \exists_{k,l \in A_n} x(k) = y(l) = 0$ or, equivalently, $(x, y) \in D \Leftrightarrow \forall_n x \restriction A_n = \mathbf{1}$ or $y \restriction A_n = \mathbf{1}$. Since it is easier to investigate properties of $D$ in $(2^\omega)^\omega \times (2^\omega)^\omega$, we will define $D$ in this space: $((x_k), (y_k)) \in D \Leftrightarrow \forall_k x_k = \mathbf{1}$ or $y_k = \mathbf{1}$, where $\mathbf{1}$ is the constant function equal to 1.

Let $E \subseteq (2^\omega)^\omega$ be a perfect set and $U_n = \{(x_k) \in (2^\omega)^\omega : x_n \neq \mathbf{1}\}$. Define $G = \bigcap_{n \in \omega} (E \cap U_n) \cup (E \setminus (\overline{E \cap U_n}))$. Let $W, V \subseteq (2^\omega)^\omega$ be open sets such that $W = W_0 \times W_1 \times \cdots \times W_{n-1} \times 2^\omega \times 2^\omega \times \cdots$ and $V = V_0 \times V_1 \times \cdots \times V_{n-1} \times 2^\omega \times 2^\omega \times \cdots$ and $(W \times V) \cap D \neq \emptyset$. We will find open sets $W^*$, $V^*$ such that $W^* \subseteq W$, $V^* \subseteq V$, and $(W^* \times V^*) \cap D \neq \emptyset$ and $(W^* \times V^*) \cap (G \times (2^\omega)^\omega) \cap D = \emptyset$. Consider the following cases:

(1) Suppose that $((W \setminus (\overline{E \cap U_n})) \times V) \cap D \neq \emptyset$. Then $W^* = (W \setminus (\overline{E \cap U_n})) \cap U_n$ and $V^* = V$. It is obvious that $(W^* \times V^*) \cap (G \times (2^\omega)^\omega) = \emptyset$. We will show that $(W^* \times V^*) \cap D \neq \emptyset$. By the assumption, there exists an open set $H = H_0 \times H_1 \times \cdots \times H_n \times \cdots \times H_k \times 2^\omega \times 2^\omega \times \cdots$ such that $(H \times V) \cap D \neq \emptyset$ and $H \subseteq W \setminus (E \cap U_n)$. It means that there exist $x = (x_k) \in H$ and $y = (y_k) \in V$ such that for every $i$, $x_i = \mathbf{1}$ or $y_i = \mathbf{1}$. Let $\overline{x}$ be an arbitrary element of $H_n$ different from $\mathbf{1}$. Then let $x^* = (x_i^*)$ such that

$$x_i^* = \begin{cases} x_i & \text{if } i \neq n, \\ \overline{x} & \text{if } i = n \end{cases}$$

and $y^* = (y_i^*)$ such that

$$y_i^* = \begin{cases} y_i & \text{if } i \neq n, \\ \mathbf{1} & \text{if } i = n. \end{cases}$$

Then $x^* \in H \cap U_n$ and $y^* \in V$, so $x^* \in W^*$ and $y^* \in V^*$ and $(x^*, y^*) \in D$ because for every $i$, $x_i^* = \mathbf{1}$ or $y_i^* = \mathbf{1}$.

(2) Suppose that $((W \setminus (\overline{E \cap U_n})) \times V) \cap D = \emptyset$. Then let $W^* = W$ and $V^* = V \cap U_n$. There exist $x = (x_i)$, $y = (y_i)$ such that $x \in W$, $y \in V$, and $(x, y) \in D$.

Let

$$x^* = (x_i^*), \quad \text{where } x_i^* = \begin{cases} x_i & \text{if } i \neq n, \\ \mathbf{1} & \text{if } i = n \end{cases}$$

and

$$y^* = (y_i^*), \quad \text{where } y_i^* = \begin{cases} y_i & \text{if } i \neq n, \\ \bar{y} & \text{if } i = n, \end{cases}$$

where $\bar{y}$ is an arbitrary element of $2^\omega$ different from $\mathbf{1}$. Then $x^* \in W = W^*$, $y^* \in V \cap U_n = V^*$, and $(x^*, y^*) \in D$.

We show that $(W^* \times V^*) \cap (G \times (2^\omega)^\omega) \cap D = \emptyset$. We know that $G \subseteq (E \cap U_n) \cup (E \setminus (\overline{E \cap U_n}))$. By the assumption, we have that $(W^* \times V^*) \cap (W \setminus \overline{E \cap U_n}) \times (2^\omega)^\omega \cap D = \emptyset$. Since $(U_n \times U_n) \cap D = \emptyset$, we have $(E \cap U_n) \times (V \cap U_n) \cap D = \emptyset$. Thus $((E \cap U_n) \times (2^\omega)^\omega) \cap (W^* \times V^*) \cap D = \emptyset$. □

*Note.* After seeing the solution to the problem given in this paper, Pawlikowski constructed an alternative solution which is more similar to the construction of Sierpinski [Si2] and uses the "function of Luzin" [L]. Pawlikowski's solution has appeared in [P].


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