PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 4, August 1991

AN EXAMPLE OF A HILBERT TRANSFORM

KRZYSZTOF SAMOTIJ

(Communicated by J. Marshall Ash)

ABSTRACT. We construct a nonnegative integrable function on the real line $\mathbf{R}$ whose Hilbert transform cannot be almost everywhere dominated by the Hardy-Littlewood maximal function of any finite measure on $\mathbf{R}$.

In this paper we construct a nonnegative integrable function on the real line $\mathbf{R}$ whose Hilbert transform cannot be almost everywhere dominated by the (Hardy-Littlewood) maximal function of any finite measure. This answers a question from [1]. The question was motivated by apparent similarities between the magnitude of maximal functions and that of Hilbert transforms as revealed, for example, by the Hardy-Littlewood maximal theorem and Kolmogorov's theorem (see e.g. [4]), Zygmunds theorem [4, Theorem 4.4], Burkholder–Gundy–Silverstein's result [3], and more recent results of Noell and Wolff [5, Proposition 0.6] (reproduced in [1, Theorem 5]) and Bruna and Korenblum [1] (see also [2]). Our present result can be compared with a complementary result of Noell and Wolff [5, p. 40] saying that if π is the product measure on the Cantor set then for each finite measure μ on the real line $\max\{0, M\pi - |H\mu|\} \notin L_{1,\mathrm{loc}}(\mathbf{R})$, where the operators M and H are defined below.

Throughout this paper let a measure be a nonnegative finite Borel measure on the real line $\mathbf{R}$. If μ is a measure then its Hilbert transform is defined (almost everywhere) as

$$H\mu(x) = \lim_{\varepsilon\to 0+} \int_{|x-y|\geq\varepsilon} \frac{d\mu(y)}{x-y}, \qquad x \in \mathbf{R}.$$

Let the maximal function of a measure μ be the function

$$M\mu(x) = \sup_{s<x<t} \frac{\mu([s, t])}{t-s}, \qquad x \in \mathbf{R}.$$

If f is a nonnegative integrable function on $\mathbf{R}$ then its Hilbert transform Hf and the maximal function Mf are defined, respectively, as the Hilbert transform and the maximal function of the measure with the density f. In what follows $|A|$ denotes the Lebesgue measure of a set A, $A \subset \mathbf{R}$.

Received by the editors July 10, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42A50.

Our construction makes use of the fact that the maximal function of a measure with the density function equal to the characteristic function of a finite nondegenerate interval multiplied by a constant is bounded by this constant while the Hilbert transform of such a measure has singularities at the endpoints of the interval. This situation will not change much if we replace such a measure by another measure that is close to it. This is expressed in some quantitative terms in the following two lemmas.

Lemma 1. *Let a number $\alpha > 0$ be arbitrary. Then there are a positive integer N and a positive number δ such that for each positive β, each real a and b, $a < b$, each integer n, $n \geq N$, and each measure σ of the form $\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_n$ with*

$$\sigma_j\left(\left[a + \frac{(j-1)(b-a)}{n}, a + \frac{j(b-a)}{n}\right]^c\right) = 0, \tag{1}$$

and

$$\sigma_j\left(\left[a + \frac{(j-1)(b-a)}{n}, a + \frac{j(b-a)}{n}\right]\right) = \frac{\beta}{n}, \tag{2}$$

$j = 1, 2, \ldots, n$, we have

$$\left|\left\{x \in (b + \delta(b-a), b + (b-a)) : H\sigma(x) \geq \frac{\alpha\beta}{b-a}\right\}\right| \geq \frac{b-a}{4e^{\alpha}}.$$

Proof. Let $\alpha > 0$ be arbitrary. It is clear that it is enough to prove the lemma in the case when $a = -1$, $b = 0$, and $\beta = 1$. Let $n \geq 2$ and suppose that a measure σ satisfies (1) and (2). Then we have

$$H\sigma_j(x) \geq H\chi_{[-1+(j-2)/n, -1+(j-1)/n]}(x), \qquad x > -1 + j/n, \ j = 1, 2, \ldots, n.$$

Adding these inequalities together we obtain:

$$H\sigma(x) \geq H\chi_{[-1-1/n, -1/n]}(x) = \log\left(\frac{x+1+1/n}{x+1/n}\right), \qquad x > 0.$$

Therefore if $n > 2e^{\alpha}$ then $H\sigma(x) \geq \alpha$ provided $0 < x \leq (2e^{\alpha})^{-1}$. This completes the proof of the lemma with $\delta = (4e^{\alpha})^{-1}$.

Lemma 2. *Let $J_1, J_2, \ldots, J_n$ be a family of open intervals, each of the length l, such that the right endpoint of J_i is the left endpoint of J_{i+1}, $i = 1, 2 \ldots, n-1$. Suppose that a measure μ is concentrated on $\bigcup_i J_i$ so that $\mu(J_i) \leq B$, $i = 1, 2, \ldots, n$, for some constant B. Then $M\mu(x) \leq B/l$ whenever $x \geq x_0 + l$, where x_0 denotes the right endpoint of J_n.*

Proof. Let $x \geq x_0 + l$ and let J be an arbitrary finite interval containing x. Let r be the number of intervals J_i with $J_i \cap J \neq \varnothing$. Then $\mu(J) = \sum_{J_i \cap J \neq \varnothing} \mu(J \cap J_i) \leq rB$. But since $x \geq x_0 + l$ we have $rl < |J|$. Therefore $M\mu(x) = \sup_{J \ni x} \mu(J)/|J| \leq B/l$.

For a nondegenerate finite closed interval $I = [a, b]$, an integer n, $n \geq 2$, and a real number θ, $0 < \theta < 1$, we define

$$\mathscr{C}(J, n, \theta) = \bigcup_{j=1}^{n} [a_j, b_j],$$

where $a = a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n = b$, $b_j - a_j = \theta|I|/n$, $j = 1, 2, \ldots, n$ and $a_2 - b_1 = a_3 - b_2 = \cdots = a_n - b_{n-1}$. If $F = \bigcup_{r=1} I_r$, where I_r's are pairwise disjoint finite nondegenerate closed intervals then we define

$$\mathscr{C}(F, n, \theta) = \bigcup_{r=1}^{p} \mathscr{C}(I_r, n, \theta).$$

For given sequences: (n_k) of integers greater than or equal to 2, (θ_k) of real numbers with $0 < \theta_k < 1$, $k = 1, 2, \ldots$, we define a decreasing sequence of closed subsets of the line $\mathbf{R}$ inductively as follows:

$$E_0 = [0, 1], \quad E_k = \mathscr{C}(E_{k-1}, n_k, \theta_k), \qquad k = 1, 2, \ldots.$$

Each E_k is the union of $n_1 \cdot n_2 \cdot \cdots \cdot n_k$ disjoint closed intervals which we denote by I_j^k, $j = 1, 2, \ldots, n_1 \cdot n_2 \cdot \cdots \cdot n_k$, where I_{j+1}^k follows after I_j^k in the ordering of the real line.

The following lemma is the core of our construction

Lemma 3. *Let (n_k) be a sequence of integers greater than or equal to 2. Let (θ_k) be a sequence of real numbers with $0 < \theta_k < 1$, $k = 1, 2, \ldots$. Suppose that $\lim_{k\to+\infty} n_k = +\infty$, and $C = \sup_k \theta_k \cdot n_1 \cdot n_2 \cdot \cdots \cdot n_k < +\infty$. Then for each positive real L there exists a positive integer K such that if ν is any measure with $\nu(E_K^c) = 0$ and $\nu(I_j^K) = (n_1 \cdot n_2 \cdots n_K)^{-1}$, $j = 1, 2, \ldots, n_1 \cdot n_2 \cdots n_K$, then for each measure μ with $\mu(\mathbf{R}) \leq L$ we have $|\{x \in (0, 2) : H\nu(x) > M\mu(x)\}| > 0$.*

Proof. Let us choose K as follows. First, let δ and K be the numbers from Lemma 1 that correspond to $\alpha = 4LC + 2L + 4C$. Then let N_1 be an integer greater than or equal to N and such that

$$n_k \geq 3/\delta + 1 \tag{3}$$

and

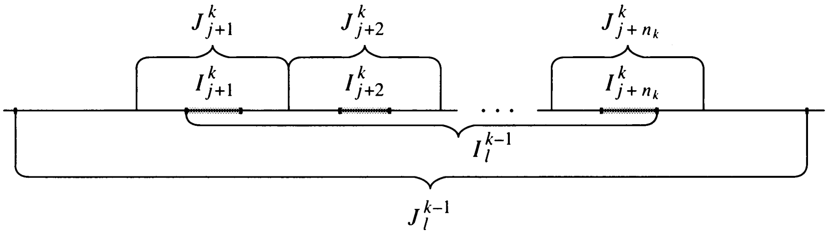
$$\theta_k \leq 1/6 \tag{4}$$

whenever $k \geq N_1$. The existence of such an N_1 follows by the assumption of the lemma. Next, let N_2 be any integer with

$$N_2 > 16e^{\alpha}. \tag{5}$$

We will show that $K = N_1 + N_2$ does the job.

For each positive integer k and each j, $j = 1, 2, \ldots, n_1 \cdot n_2 \cdots n_k$, let J_j^k be an interval concentric with I_j^k, chosen so that, for each fixed k, the lengths of J_j^k's are all equal to the minimum distance between the centers of



I^k_j's. Therefore $J^k_j \cap J^k_i = \varnothing$ whenever $j \neq i$. Note that if I^k_j and I^k_{j+1} are subsets of the same I^{k-1}_i, then the right endpoint of J^k_j coincides with the left endpoint of J^k_{j+1}. Note also that

$$(6) \qquad |I^k_j| = \frac{|E_k|}{n_1 \cdots n_k} = \frac{\theta_1 \cdots \theta_k}{n_1 \cdots n_k}.$$

The length of J^k_j equals $(|I^{k-1}_i| - |I^k_j|)/(n_k - 1)$, so, by (6), we have

$$(7) \qquad |J^k_j| = \frac{\theta_1 \cdots \theta_{k-1}}{n_1 \cdots n_k} \frac{n_k - \theta_k}{n_k - 1}.$$

It is easy to see that each J^{k+1}_j is contained on some J^k_i.

Let μ be any measure with $\mu(\mathbf{R}) \leq L$. Let a sequence $(j(k))$ be chosen inductively so that

$$(8) \qquad \mu(J^1_{j(1)}) = \min\{\mu(J^1_i) : i = 1, 2, \ldots, n_1\},$$

and

$$(9) \qquad J^k_{j(k)} \subset J^{k-1}_{j(k-1)}, \qquad \mu(J^k_{j(k)}) = \min\{\mu(J^k_i) : J^k_i \subset J^{k-1}_{j(k-1)}\}, \ k \geq 2.$$

Since J^1_j's are pairwise disjoint we have, by (8), $n_1\mu(J^1_{j(1)}) \leq \mu(R) \leq L$. Similarly, (9) implies that $n_k\mu(J^k_{j(k)}) \leq \mu(J^{k-1}_{j(k-1)})$, $k \geq 2$. Hence the nonnegative sequence $(n_1 \cdots n_k\mu(J^k_{j(k)}))$ is nonincreasing and bounded by L from above. Therefore there is a $\tilde{k}$, $N_1 < \tilde{k} \leq N_1 + N_2 = K$, such that

$$(10) \qquad n_1 \cdots n_{\tilde{k}-1}\mu(J^{\tilde{k}-1}_{j(\tilde{k}-1)}) - n_1 \cdots n_{\tilde{k}}\mu(J^{\tilde{k}}_{j(\tilde{k})}) \leq L/N_2.$$

To simplify the notation we denote $\widetilde{I} = [a, b] = I^{\tilde{k}-1}_{j(\tilde{k}-1)}$ and $\widetilde{J} = (c, d) = J^{\tilde{k}-1}_{j(\tilde{k}-1)}$. Now we will obtain some estimates of $M\mu(x)$ for $x \in (b + \delta(b-a), b + (b-a))$. First, observe that for such an x, by (4), we have

$$(11) \qquad M(\mu|_{\widetilde{J}^c})(x) \leq \frac{\mu(\mathbf{R})}{\operatorname{dist}(x, \widetilde{J}^c)} = \frac{\mu(\mathbf{R})}{d - x} \leq \frac{L}{d - [b + (b-a)]} = \frac{L}{|\widetilde{J}|/2 - (3/2)|\widetilde{I}|}.$$

Therefore, by (6), (7), (4), and the assumption of the lemma, we have

$$M(\mu|_{\tilde{J}^c}) \leq \frac{4LC}{\theta_1 \cdots \theta_{\tilde{k}-1}}, \qquad b+\delta(b-a) < x < b+(b-a). \tag{12}$$

In order to estimate $M(\mu|_{\tilde{J}})(x)$ denote $\mathcal{J} = \{J_j^{\tilde{k}} : J_j^{\tilde{k}} \subset \tilde{J}\}$ and write $\mu|_{\tilde{J}} = \mu_1 + \mu_2$, where μ_1 and μ_2 are measures chosen so that $\mu_1((\cup\mathcal{J})^c) = 0$ and $\mu_1(J) = \min\{\mu(J) : J \in \mathcal{J}\} = \mu(J_{j(\tilde{k})}^{\tilde{k}})$ for each $J \in \mathcal{J}$. Lemma 2 applied to $\mathcal{J}$ and μ_1 gives, by (6),

$$\begin{aligned} M\mu_1(x) &\leq \frac{\mu_1(J_{j(\tilde{k})}^{k})}{|J_{j(\tilde{k})}^{\tilde{k}}|} = \frac{n_1 \cdots n_{\tilde{k}}\mu(J_{j(\tilde{k})}^{\tilde{k}})}{n_1 \cdots n_{\tilde{k}}|J_{j(\tilde{k})}^{\tilde{k}}|} \\ &\leq \frac{L}{n_1 \cdots n_{\tilde{k}-1}|\tilde{I}|} = \frac{L}{\theta_1 \cdots \theta_{\tilde{k}-1}} \end{aligned} \tag{13}$$

if

$$x \geq \sup(\cup\mathcal{J}) + |J_j^{\tilde{k}}| = b + \tfrac{3}{2}|J_j^{\tilde{k}}| - \tfrac{1}{2}|I_j^{\tilde{k}}|. \tag{14}$$

But (6), (7), and (3) imply that the right-hand side of (14) does not exceed $b + \delta|\tilde{I}| = b + \delta(b-a)$. Hence, by (13) and (14), we have

$$M\mu_1(x) \leq \frac{L}{\theta_1 \cdots \theta_{\tilde{k}-1}}, \qquad x \in (b+\delta(b-a), b+(b-a)). \tag{15}$$

To deal with $M\mu_2(x)$ let us note that, by (10),

$$\mu_2(\mathbf{R}) = \mu(\tilde{J}) - n_{\tilde{k}}\mu(J_{j(\tilde{k})}^{\tilde{k}}) \leq \frac{L}{N_2 n_1 n_2 \cdots n_{\tilde{k}-1}}.$$

Therefore by the Hardy-Littlewood maximal theorem, and by (5) and (6), we have

$$\begin{aligned} \left|\left\{x \in \mathbf{R} : M\mu_2(x) \geq \frac{L}{\theta_1 \cdots \theta_{\tilde{k}-1}}\right\}\right| &\leq \frac{2\mu_2(\mathbf{R})\theta_1 \cdots \theta_{\tilde{k}-1}}{L} \\ &\leq \frac{\theta_1 \cdots \theta_{\tilde{k}-1}}{8e^{\alpha} n_1 \cdots n_{\tilde{k}-1}} = \frac{|\tilde{I}|}{8e^{\alpha}}. \end{aligned} \tag{16}$$

Since $M\mu \leq M\mu_1 + M\mu_2 + M(\mu|_{\tilde{J}^c})$, combining (12), (15) and (16) we obtain

$$\left|\left\{x \in (b+\delta(b-a), b+(b-a)) : M\mu(x) \geq \frac{4LC+2L}{\theta_1 \cdots \theta_{\tilde{k}-1}}\right\}\right| \leq \frac{|\tilde{I}|}{8e^{\alpha}}.$$

Therefore to complete the proof it is enough to show that

$$\left|\left\{x \in (b+\delta(b-a), b+(b-a)) : H\nu(x) \geq \frac{4LC+2L}{\theta_1 \cdots \theta_{\tilde{k}-1}}\right\}\right| > \frac{|\tilde{I}|}{8e^{\alpha}}. \tag{17}$$

To this end we split $H\nu$ into the sum of $H(\nu|_{\tilde{I}^c})$ and $H(\nu|_{\tilde{I}})$. The first term can be estimated similarly as $M(\mu|_{\tilde{J}})$ was in (11) and (12). Namely, if $b+\delta(b-a) < x < (b-a)$ then

$$H(\nu|_{\tilde{I}^c})(x) = H(|\nu|_{\tilde{J}^c})(x) \geq \frac{-\nu(\mathbf{R})}{\operatorname{dist}(x, \tilde{J}^c)} \geq \frac{-4C}{\theta_1 \cdots \theta_{\tilde{k}-1}}. \tag{18}$$

To estimate $H(\nu|_{\tilde{I}})$ note that the measure $\nu|_{\tilde{I}}$ satisfies the assumptions of Lemma 1 applied to the interval $\tilde{I}$ with $\beta = (n_1 \cdots n_{k-1})^{-1}$ and $n = n_k$. Hence that lemma together with (6), for $\alpha = 4LC + 2L + 4C$, give

$$\left|\left\{x \in (b+\delta(b-a), b+(b-a)) : H(\nu|_{\tilde{I}})(x) \geq \frac{\alpha}{\theta_1 \cdots \theta_{\tilde{k}-1}}\right\}\right| \geq \frac{|\tilde{I}|}{4e^\alpha} > \frac{|\tilde{I}|}{8e^\alpha}. \tag{19}$$

The inequalities (18) and (19) imply (17).

Lemma 3 implies the following fact which seems interesting enough to call it a theorem.

Theorem 1. *Let sequences (n_k) and (θ_k) satisfy the assumption of Lemma 3. Let π be the product measure on the Cantor-type set $E = \bigcap_{k=1}^\infty E_k$ associated with these sequences, that is π is a probability measure supported on E such that for each k all the components of the set E_k have the same π measure. Then*

$$|\{x \in (0, 2) : H\pi(x) > M\mu(x)\}| > 0$$

for each measure μ.

To prove our main theorem let us introduce an auxiliary notation. For each finite closed interval I of the line $\mathbf{R}$ and each real function g defined on I let us set

$$L_I(g) = \inf\{\mu(R) : \mu \text{ is a measure with } M\mu \geq g \text{ a.e. on } I\},$$

where we assume that $\inf\emptyset = +\infty$. Note that in the definition of $L_I(g)$ we can restrict ourselves to measures concentrated on I since for each measure μ we have $M\mu(x) \leq M(\mu|_I + \mu((-\infty, a))\delta_a + \mu((b, +\infty))\delta_b)(x)$, $x \in I$, where a and b are, respectively, the left and the right endpoints of I, and δ_a and δ_b are the Dirac deltas.

Lemma 4. *For any nondegenerate finite closed interval $I = [a, b]$, and any real g on I, and any positive constant C we have*

$$L_I(g - C) \geq L_I(g) - C|I|.$$

Proof. Let λ be a measure defined by $\lambda(A) = |A \cap I|$, for each Borel subset A of $\mathbf{R}$. For each measure μ concentrated on I, and each $x \in (a, b)$ we have

$$(20)\qquad \begin{aligned} M(\mu + C\lambda)(x) &= \sup_{a \leq s < x < t \leq b} \frac{(\mu + C\lambda)([s, t])}{t - s} \\ &= \sup_{a \leq s < x < t \leq b} \frac{\mu([s, t])}{t - s} + C = M\mu(x) + C. \end{aligned}$$

Let us suppose that $L_I(g - C) < L_I(g) - C|I|$. Then there is a measure μ concentrated on I such that $\mu(\mathbf{R}) < L_I(g) - C|I|$ and $M\mu \geq g - C$ a.e. on I. But, by (20), $M(\mu + C\lambda) = M\mu + C \geq g$ a.e. on I, and $(\mu + C\lambda)(\mathbf{R}) = \mu(\mathbf{R}) + C|I| < L_I(g)$. This contradicts the definition of $L_I(g)$.

Theorem 2. *There is a nonnegative integrable function* g *on* $\mathbf{R}$, *vanishing outside* $(0, 1)$ *such that* $|\{x \in [0, 1] : Hg(x) > M\mu(x)\}| > 0$ *for each measure* μ.

Proof. If we fix sequences (n_k) and (θ_k) satisfying the assumption of Lemma 3 (e.g. $n_k = k + 1$ and $\theta_k = 1/(k + 1)!$) and set $f_k = \chi_{E_k}/|E_k|$, $k = 1, 2, \ldots$, then $\int f_k = 1$, and, by Lemma 3, $\lim_{k \to +\infty} L_{[0, 2]}(Hf_k) = +\infty$. It is not hard to see that for any nondegenerate finite closed interval $I = [a, b]$, and arbitrary positive numbers ε and L there exists a nonnegative integrable function f on $\mathbf{R}$, vanishing outside I such that $\int f = \varepsilon$ and $L_I(Hf) \geq L$. To construct such a function it is enough to take $f = \varepsilon(f_k \circ \tau)/|I|$ for large enough k, where τ is a linear function with $\tau(a) = 0$ and $\tau(b) = 2$.

Let $I_n = [2^{-n}, 3 \cdot 2^{-n-1}]$, $n = 1, 2, \ldots$. Note that $\operatorname{dist}(I_n, I_{n-1}) = 2^{-n-1}$, $n = 2, 3, \ldots$. For each positive integer n, let g_n be a nonnegative integrable function on $\mathbf{R}$, vanishing outside I_n such that $\int g_n = 2^{-n}$ and

$$(21)\qquad L_{I_n}(Hg_n) \geq n.$$

Let $g = \sum_{n=1}^{\infty} g_n$. The function g is nonnegative, vanishes outside $(0, 1)$, and $\int g = 1$. For each $n \geq 2$ we have

$$(22)\qquad \begin{aligned} L_{[0, 1]}(Hg) &\geq L_{I_n}\left(Hg_n + H\left(\sum_{m \neq n} g_m\right)\right) \\ &\geq L_{I_n}\left(Hg_n + \inf_{x \in I_n} H\left(\sum_{m \neq n} g_m\right)\right). \end{aligned}$$

Let $x \in I_n$ be arbitrary. Since $g_m(y) = 0$ whenever $y \geq x$ and $m > n$, we have

$$(23)\qquad H\left(\sum_{m > n} g_m\right)(x) = \int_{-\infty}^{x} \left(\sum_{m < n} \frac{g_m(y)}{x - y}\right) dy \geq 0.$$

On the other hand if $x \in I_n$ then

$$(24)\qquad \begin{aligned} H\left(\sum_{m < n} g_m\right)(x) &\geq -\int \left(\sum_{m < n} g_m\right) \Big/ \operatorname{dist}(x, I_{n-1}) \\ &\geq -1/\operatorname{dist}(I_n, I_{n-1}) = -2^{n+1}. \end{aligned}$$

Combining (23) and (24) we obtain

$$\inf_{x\in I_n} H\left(\sum_{m\neq n} g_m\right)(x) \geq -2^{n+1}.$$

Therefore, by (22), Lemma 4, and (21), we have

$$\begin{aligned} L_{[0,1]}(Hg) &\geq L_{I_n}(Hg_n - 2^{n+1}) \geq L_{I_n}(Hg_n) - 2^{n+1}|I_n| \\ &= L_{I_n}(Hg_n) - 1 \geq n-1. \end{aligned}$$

Since n was arbitrary we have $L_{[0,1]}(Hg) = +\infty$, which is all we need to complete the proof.

Acknowledgment

This work was done while the author was visiting the Department of Mathematics of the State University of New York at Albany. He wishes to thank members of the faculty of this institution, especially Boris Korenblum, for hospitality and valuable discussions.

References

1. J. Bruna and B. Korenblum, *On Kolmogorov's theorem, the Hardy-Littlewood maximal function and the radial maximal function*, J. Analyse Math. **50** (1988), 225–239.
2. ——, *A note on Calderon-Zygmund singular integral convolution operators*, Bull. Amer. Math. Soc. (N.S.) **16** (1987), 271–273.
3. D. L. Burkholder, R. F. Gundy and M. L. Silverstein, *A maximal function characterization of the class* H^p, Trans. Amer. Math. Soc. **157** (1971), 137–153.
4. P. L. Duren, *Theory of* H^p *spaces*, Academic Press, New York and London, 1970.
5. A. V. Noell and T. H. Wolff, *On peak sets for* Lip α *classes*, J. Funct. Anal. **86** (1989), 136–179.

Instytut Matematyki, Politechnika Wrocławska, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland