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VON NEUMANN ALGEBRAS WHICH ARE SECOND DUAL SPACES

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ABSTRACT. A σ-finite von Neumann algebra is a second dual if and only if it is atomic.

Shtern [7] has shown that a C^*-algebra is a third dual if and only if it is the enveloping von Neumann algebra of a von Neumann algebra. C^*-algebras that are dual spaces are the von Neumann algebras. The remaining question appears to be: when is a C^*-algebra a second dual? This question is the same as asking when a von Neumann algebra is a second dual. Let M be a von Neumann algebra and let $M = X^{**}$ for some Banach space X. By the Krein-Milman Theorem, M has a pure normal state and hence M has a type I summand. Can M have type II and type III summands as well? The answer is yes if, for instance, X is the quotient of the algebra $B(H)$ of bounded operators on a separable Hilbert space H by the compact operators $K(H)$ [1, 6]. The above observation seems to suggest, without some restriction, that the question may be too general to have a fruitful answer. If, however, we restrict the question to the class of σ-finite von Neumann algebras, then a simple answer emerges.

Theorem. *A σ-finite von Neumann algebra M is a second dual space if and only if M is atomic, that is, if and only if M is a direct sum of the $B(H)$'s.*

In the commutative case, the above is of course a well-known result of Pelczynski [4, Theorem 4] since a commutative σ-finite von Neumann algebra is just an $L_\infty(\mu)$ where μ is a σ-finite measure and in this case, Pelczynski has shown that $L_1(\mu)$ is (isomorphic to) a dual if and only if μ is purely atomic. Rosenthal [5, Corollary 2.2] has given a short proof of this result. Distel and Uhl [3, p. 83] gave a shorter proof using the Radon-Nikodym property and their approach works well in the noncommutative case.

Recall that the σ-finite von Neumann algebras are those that admit faithful normal states. Examples: (i) any von Neumann algebra acting on a separable Hilbert space; (ii) all von Neumann algebras appeared in quantum statistical mechanics and quantum field theory. A Banach space is called *weakly compactly generated* (WCG) if it is the (norm) closed linear span of one of its

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weakly compact subsets. Infinite-dimensional von Neumann algebras are not WCG since a WCG von Neumann algebra has the Radon-Nikodym property [3, p. 87] and must be finite dimensional.

Lemma. *A von Neumann algebra M is σ-finite if and only if its predual M_* is WCG.*

Proof. Suppose M has a faithful normal state $\varphi \in M_*$. By [8, Proposition 5.12], the set $K = \{\varphi(a.): a \in M, \|a\| \leq 1\}$ is a relatively $\sigma(M_*, M)$-compact set in M_*. The linear span $\operatorname{lin} K$ is (norm) dense in M_*. Otherwise, there exists nonzero b in M with $\|b\| \leq 1$ such that $\langle \psi, b\rangle = 0$ for all $\psi \in \operatorname{lin} K$. If $b = u|b|$ is the polar decomposition, then $\langle \varphi, |b|\rangle = \langle \varphi(u^*.), b\rangle = 0$ as $\varphi(u^*.) \in K$, contradicting the faithfulness of φ. So M_* is WCG. Conversely, let M_* be generated by a $\sigma(M_*, M)$-compact set S. By [8, Theorem 5.4], there is a normal state $\omega \in M_*$ with the property that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\langle \psi, a\rangle| < \varepsilon$ for all $\psi \in S$ whenever $\|a\| \leq 1$ and $\langle \omega, a^*a + aa^*\rangle < \delta$. Hence $\langle \omega, a^*a\rangle = 0$ implies $\langle \psi, (a^*a)^{1/2}\rangle = 0$ for all $\psi \in S$ and so $\langle \psi, (a^*a)^{1/2}\rangle = 0$ for all $\psi \in M_*$, giving $a = 0$. That is, ω is faithful.

Now, if a von Neumann algebra M is σ-finite as well as a second dual, then its predual M_* is WCG and a dual. So M_* has the Radon-Nikodym property [3, p. 87]. By [2, Theorem 4], M is atomic.

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