PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 4, August 1991

A COUNTEREXAMPLE FOR KOBAYASHI COMPLETENESS OF BALANCED DOMAINS

MAREK JARNICKI AND PETER PFLUG

(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. The aim of this paper is to present an example of a bounded balanced domain of holomorphy in $\mathbb{C}^n$ $(n \geq 3)$ with continuous Minkowski function that is not Kobayashi-finitely-compact.

INTRODUCTION

It is known [6] that if $G \subset \mathbb{C}^n$ is a bounded Reinhardt-domain of holomorphy with $0 \in G$ then G is finitely-compact with respect to (w.r.t.) the Carathéodory-distance c_G, i.e., all c_G-balls are relatively compact subsets of G w.r.t. the usual topology. In a more general case, if $G = G_h = \{z \in \mathbb{C}^n : h(z) < 1\}$ is a bounded balanced domain of holomorphy with continuous Minkowski function h, then G is finitely-compact w.r.t. the Bergman-distance b_G [4]. On the other hand, the continuity of h is a necessary condition for $G = G_h$ to be finitely-compact w.r.t. the Kobayashi-distance k_G [5, 1].

In this paper we give an example of a bounded balanced domain of holomorphy $G = G_h \subset \mathbb{C}^3$ with continuous h that is not k_G-finitely compact and therefore, not c_G-finitely compact. This answers a question formulated by J. Siciak in [7].

In particular, the example shows that, in general, there is no comparison of type $b_G \leq Ck_G$ for bounded balanced domains of holomorphy with continuous Minkowski function.

DEFINITIONS AND STATEMENT

We repeat some of the notions that will be needed in the sequel.

Definition. A domain $G \subset \mathbb{C}^n$ is called *balanced*[1] iff whenever $z \in G$ and $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$ then $\lambda z \in G$.

If $G \subset \mathbb{C}^n$ is balanced then there exists a unique homogeneous function

Received by the editors October 6, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32H15.

[1]Often such a domain is also called *complete circular*.

$h: \mathbb{C}^n \to [0, \infty]$ $(h(\lambda z) = |\lambda| h(z),\ \lambda \in \mathbb{C},\ z \in \mathbb{C}^n)$ such that

$$G = G_h = \{z \in \mathbb{C}^n : h(z) < 1\}.$$

The function h is called the *Minkowski function* of G.

It is known that G_h is a bounded balanced domain of holomorphy if h is plurisubharmonic and $h^{-1}(0) = \{0\}$.

As in the Introduction, if G is any domain in $\mathbb{C}^n$ we denote by k_G the Kobayashi-distance on G i.e. for $z', z'' \in G$,

$$k_G(z', z'') := \inf\left\{ \sum_{j=1}^{k} \tilde{k}_G(z_{j-1}, z_j) : k \in \mathbb{N} \text{ and } z_0 = z', z_1, \dots, z_k = z'' \in G \right\}$$

where, if $w', w'' \in G$ then

$$\tilde{k}_G(w', w'') = \inf\{\rho_E(\zeta', \zeta'') : \zeta', \zeta'' \in E \text{ and } f: E \to G \text{ holomorphic and } f(\zeta') = w',\ f(\zeta'') = w''\}$$

with $E := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and

$$\rho_E(\lambda', \lambda'') = \frac{1}{2} \log \left(1 + \left|\frac{\lambda' - \lambda''}{-\bar{\lambda}''\lambda' + 1}\right|\right) \Big/ \left(1 - \left|\frac{\lambda' - \lambda''}{-\bar{\lambda}''\lambda' + 1}\right|\right)$$

being the Poincaré-distance (hyperbolic distance) on E. Observe that: $c_G \le k_G$ and $c_G \le b_G$ [5, 3].

Definition. A bounded domain $G \subset \mathbb{C}^n$ is called *k_G-finitely-compact* if, for all $R > 0$ and all $z^0 \in G$, the k_G-ball

$$\{z \in G : k_G(z^0, z) < R\}$$

is a relatively compact subset of G w.r.t. the usual topology.

Remark. It is known that the finitely-compactness for k_G coincides with the completeness of the metric space (G, k_G).

Observe that c_G-finitely-compactness implies finitely-compactness w.r.t. b_G and w.r.t. k_G.

The aim of this paper is to prove the following result:

Theorem. *There exists a bounded balanced domain of holomorphy $G = \{z \in \mathbb{C}^3 : h(z) < 1\}$ in $\mathbb{C}^3$ with continuous Minkowski function h that is not k_G-finitely-compact.*

Observe that this G is not complete w.r.t. c_G if we regard (G, c_G) to be a metric space.

For $n \ge 3$ the domain $D = G \times E^{n-3}$ gives an analogous example in $\mathbb{C}^n$.

Remark. The construction of the example will be based on the ideas of N. Sibony [private communication, (1981)] that he used to construct a bounded domain of holomorphy $D \subset \mathbb{C}^2$ such that ∂D is smooth and strictly pseudoconvex except of one point, but D is not k_D-finitely-compact.

Construction of the example

In order to construct our example we first present two general reduction steps:

(A) Let $g: \mathbb{C}^n \to \mathbb{R}_+ := [0, \infty)$ be a continuous, logarithmically-plurisubharmonic (log-psh) function with the following two properties:

(1) $l := \lim_{\|z\| \to \infty} g(z)/\|z\|$ exists with $l \in \mathbb{R}_+$;

(2) $G := \{z \in \mathbb{C}^n : g(z) < 1\}$ is bounded and has a connected component G' that is not $k_{G'}$-finitely-compact.

From these assumptions we are going to construct a bounded balanced domain of holomorphy $\widetilde{G}$ in $\mathbb{C}^{n+1}$ with continuous Minkowski function that is not $k_{\widetilde{G}}$-finitely-compact.

Define $h_0: \mathbb{C}^n \times \mathbb{C} \to [0, \infty)$ by

$$h_0(z, z_{n+1}) := \begin{cases} |z_{n+1}| \cdot g(z/z_{n+1}) & \text{if } z_{n+1} \neq 0, \\ l \cdot \|z\| & \text{if } z_{n+1} = 0. \end{cases}$$

Observe that the function h_0 is homogeneous on $\mathbb{C}^{n+1}$, continuous and log-psh on $\mathbb{C}^n \times (\mathbb{C} \setminus \{0\})$.

Moreover by (1), h_0 is continuous on the whole of $\mathbb{C}^{n+1}$. Therefore, by a theorem of Grauert–Remmert on removable singularities [2], it follows that h_0 is a continuous, plurisubharmonic, and homogeneous function on $\mathbb{C}^{n+1}$.

By assumption (2) we know that for some $r_0 \in (0, \infty)$, $G \subset B(r_0) := \{z \in \mathbb{C}^n : \|z\| < r_0\}$. Now let $h: \mathbb{C}^n \times \mathbb{C} \to \mathbb{R}_+$ be defined as

$$h(z, z_{n+1}) := \max\left(h_0(z, z_{n+1}), \sqrt{\frac{\|z\|^2 + |z_{n+1}|^2}{r_0^2 + 1}}\right).$$

Then h is a continuous, plurisubharmonic, and homogeneous function on $\mathbb{C}^{n+1}$ with $h^{-1}(0) = \{0\}$.

Thus the domain

$$\widetilde{G} := \{(z, z_{n+1}) \in \mathbb{C}^n \times \mathbb{C} : h(z, z_{n+1}) < 1\}$$

is a bounded balanced domain of holomorphy and $G' \times \{1\}$ is a connected component of

$$\widetilde{G} \cap (\mathbb{C}^n \times \{1\}) = G \times \{1\}.$$

Therefore, $\widetilde{G}$ is not $k_{\widetilde{G}}$-finitely-compact.

(B) Assume there is a connected set $X \subset B(r_0) \subset \mathbb{C}^n$ and a sequence $\{z^j\} \subset X$, $z^j \to 0$, with the following properties:

(1) There are holomorphic maps $f_j: E \to \mathbb{C}^n$, such that: $f_j(E) \subset X$,

$$f_j(\lambda_j') = z^j\,,\ f_j(\lambda_j'') = z^{j+1}\ (\lambda_j', \lambda_j'' \in E)\,;$$

$$\sum_{j=1}^{\infty} \rho_E(\lambda_j', \lambda_j'') =: A < \infty\,;$$

(2) There exists $\varphi\colon \mathbb{C}^n \to \mathbb{R}_+$ continuous and log-psh such that:
(a) $\varphi(0) = 1$, $\varphi|_X < 1$;
(b) for suitable C, α, R positive, the following inequality is true: $\varphi(z) \le C\|z\|^\alpha$ if $\|z\| \ge R$.

From these assumptions we construct g as in (A) as follows: Taking $\varphi^{1/2\alpha}$ instead of φ we can assume that $\alpha = 1/2$. Then define the function $g\colon \mathbb{C}^n \to \mathbb{R}_+$ as

$$g(z) := \max(\varphi(z), \|z\|/2r_0)$$

Of course, g is a continuous log-psh function on $\mathbb{C}^n$ with $\lim_{\|z\|\to\infty} g(z)/\|z\| = 1/2r_0$, $G := \{z \in \mathbb{C}^n\colon g(z) < 1\} \subset B(2r_0)$, and the connected component G' of G with $X \subset G'$ is not $k_{G'}$-finitely-compact because $k_{G'}(z^1, z^j) \le \sum_{\nu=1}^{j} \rho_E(\lambda_\nu', \lambda_\nu'') \le A$ and $z^j \to 0 \in \partial G'$.

(C) Now we use a part of Sibony's construction in order to reach the situation of (B): Let $c_j := 1/2^{j+1}$ and let $a_j := 1/2^{2(j+1)}$; we define

$$X_j := \{(z, w) \in \mathbb{C}^2\colon w = z(a_{j+1} + a_j) - a_j a_{j+1}\,,\ |z| \le c_j\}\,.$$

Then X_j is a connected, compact set with $X_j \subset B(1)$ and $X_j \underset{j\to\infty}{\to} 0$. Moreover, for $z^j := (a_j, a_j^2)$, we have: $z^j, z^{j+1} \in X_j$ and if $f_j\colon E \to \mathbb{C}^2$ is the holomorphic map given by $f_j(\lambda) := (c_j\lambda, c_j\lambda(a_j + a_{j+1}) - a_j a_{j+1})$, then $f_j(E) \subset X_j$, $f_j(\lambda_j') = z^j$, $f_j(\lambda_j'') = z^{j+1}$ where $\lambda_j' = a_j/c_j$, $\lambda_j'' = a_{j+1}/c_j \in E$. Direct calculations give: $\sum_{j=1}^{\infty} \rho_E(\lambda_j', \lambda_j'') < +\infty$. Let $X := \bigcup_{j\ge1} X_j$. Thus the properties (1) of (B) are satisfied.

In order to construct the function φ as in (B) set:

$$P_j(z, w) := w - z(a_j + a_{j+1}) + a_j a_{j+1}\,,$$

and observe that:

$$|P_j(\xi)| \le 3\|\xi\| \quad \text{whenever } \|\xi\| \ge 1\,,\ \xi = (z, w)\,.$$

Next we choose sequences $\{r_j\}$ and $\{\varepsilon_j\}$ of positive numbers with the following properties:

$$r_j \ge 3\,, \qquad r_j \ge 2\max\{|P_j(\xi)|\colon \|\xi\| \le j\}\,,$$

$$\log\frac{3}{4} = \sum_{j=1}^{\infty} \varepsilon_j \log\frac{a_j a_{j+1}}{r_j}\,.$$

In particular: $\log\frac{4}{3} \ge \sum_{j=1}^{\infty} \varepsilon_j =: \alpha$.

Using these sequences we obtain the following plurisubharmonic function $\Psi\colon \mathbb{C}^2 \to [-\infty, \infty)$:

$$\Psi(\xi) := \sum_{j=1}^{\infty} \varepsilon_j \log \frac{|P_j(\xi)|}{r_j}.$$

Note that $\Psi(0) = \log \frac{3}{4}$, $\Psi|_X = -\infty$,

$$\Psi < 0 \quad \text{on } \overline{B(1)}$$

and

$$\Psi(\xi) \le \alpha \log \|\xi\|, \quad \text{if } \|\xi\| \ge 1.$$

Let $\Phi\colon \mathbb{C}^2 \to \mathbb{R}_+$ with $\Phi \in C_0^\infty(\mathbb{C}^2)$, $\Phi(z, w) = \Phi(|z|, |w|)$, and with $\int_{\mathbb{C}^2} \Phi(\xi)\, d\lambda(\xi) = 1$. Setting for $\xi \in \mathbb{C}^2$, $\varepsilon > 0$,

$$\Psi_\varepsilon(\xi) := \int_{\mathbb{C}^2} \Psi(\xi - \varepsilon\eta)\Phi(\eta)\, d\eta$$

we obtain a sequence of plurisubharmonic C^∞-functions on $\mathbb{C}^2$ with

$$\Psi_\varepsilon \underset{\varepsilon \searrow 0}{\searrow} \Psi.$$

In particular, $\Psi_\varepsilon(0) \ge \log \frac{3}{4}$ and $\Psi_\varepsilon(\xi) < 0$ if $\|\xi\| \le 1$, $0 < \varepsilon \le \varepsilon_0 < 1$. By construction, if $0 < \varepsilon \le 1$, $\|\xi\| \ge 2$, then

$$\Psi_\varepsilon(\xi) \le \alpha \log(2\|\xi\|).$$

Now we define, for $0 < \varepsilon \le \varepsilon_0$:

$$\tilde{\varphi}_\varepsilon(\xi) := \frac{\exp \Psi_\varepsilon(\xi)}{\exp \Psi_\varepsilon(0)}.$$

We have obtained a sequence of positive, log-psh functions on $\mathbb{C}^2$ such that:

$$\begin{aligned} &\tilde{\varphi}_\varepsilon(0) = 1, \qquad \tilde{\varphi}_\varepsilon(\xi) < \tfrac{4}{3} \quad \text{if } \|\xi\| \le 1, \\ &\tilde{\varphi}_\varepsilon(\xi) \le C\|\xi\|^\alpha \qquad\qquad\quad \text{if } \|\xi\| \ge 2, \end{aligned}$$

where C is a positive constant independent of ε.

Now we follow the procedure of Bishop's construction of peak-functions: Let $U_1 := B(\frac{1}{2})$. Choose $\varepsilon_1 \in (0, \varepsilon_0)$ such that $\varphi_1(\xi) := \tilde{\varphi}_{\varepsilon_1}(\xi)$ is less than $\frac{1}{3}$ on $X \setminus U_1$.

Suppose we have already constructed neighborhoods of the origin:

$$U_1 \supset U_2 \supset \cdots \supset U_k \quad \text{and numbers } \varepsilon_1 > \cdots > \varepsilon_k$$

such that with $\varphi_j := \tilde{\varphi}_{\varepsilon_j}$, we have:

$$\varphi_j(\xi) < \frac{1}{3} \quad \text{on} \quad X \setminus U_j \qquad (1 \le j \le k)$$

and

$$\varphi_\nu(\xi) < 1 + \frac{1}{3}2^{-j} \quad \text{on} \quad U_j \qquad (1 < j \le k \text{ and } 1 \le \nu < j).$$

Now define

$$U_{k+1} := \{\xi \in U_k : \varphi_j(\xi) < 1 + 1/(3 \cdot 2^{k+1}),\ 1 \le j \le k\}$$

and choose $\varepsilon_{k+1} \in (0, \varepsilon_k)$ in such a way that the function

$$\varphi_{k+1} := \tilde{\varphi}_{\varepsilon_{k+1}} \quad \text{is less than } \tfrac{1}{3} \text{ on } X \setminus U_{k+1}.$$

Using the constructed sequence $\{\varphi_j\}$ we set, for $\xi \in \mathbb{C}^2$,

$$\varphi(\xi) := \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi_j(\xi).$$

The series is locally uniformly convergent and therefore, $\varphi: \mathbb{C}^2 \to \mathbb{R}_+$ is a continuous log-psh function on $\mathbb{C}^2$ with $\varphi(0) = 1$ which satisfies

$$\varphi(\xi) \le C\|\xi\|^{\alpha}, \quad \text{if } \|\xi\| \ge 2.$$

Moreover, for $\xi \in X$, we have

1. case: $\xi \in X \setminus U_1 \Rightarrow \varphi(\xi) < 1$;
2. case: $\xi \in X \cap (U_k \setminus U_{k+1}) \Rightarrow \varphi(\xi) \le 1 - 1/(3 \cdot 2^{2k-1}) < 1$.

Thus the function φ satisfies all the properties needed in part (B). Therefore, combining the steps (A), (B), and (C), we get our example.

Acknowledgment

We would like to express our gratitude to the Jagiellonian University (Kraków) and the University of Osnabrück (Vechta) for their hospitality during our research work.

References

1. T. J. Barth, *The Kobayashi indicatrix at the center of a circular domain*, Proc. Amer. Math. Soc. **88** (1983), 527–530.
2. H. Grauert and R. Remmert, *Plurisubharmonische Funktionen in komplexen Räumen*, Math. Z. **65** (1956), 175–194.
3. K. T. Hahn, *On completeness of the Bergman metric and its subordinate metrics.* II, Pacific J. Math. **68** (1977), 437–446.
4. M. Jarnicki and P. Pflug, *Bergman-completeness of complete circular domains*, Ann. Polon. Math. **50** (1989), 103–106.
5. S. Kobayashi, *Intrinsic distances, measures and geometric function theory*, Bull. Amer. Math. Soc. **82** (1976), 357–416.
6. P. Pflug, *About the Carathéodory completeness of all Reinhardt domains*, Functional Analysis, Holomorphy and Approximation Theory II (G. I. Zapata, ed.), North-Holland Amsterdam, 1984, pp. 331–337.
7. J. Siciak, *Balanced domains of holomorphy of type* H^{∞}, Mat. Vesnik **37** (1985), 134–144.

Uniwersytet Jagielloński, Instytut Matematyki, 30-059 Kraków, Reymonta 4, Poland

Universität Osnabrück, Standort Vechta, Driverstr. 22, 2848 Vechta, Germany