PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 4, August 1991

ON THE EXISTENCE AND UNIQUENESS OF FIXED POINTS FOR HOLOMORPHIC MAPS IN COMPLEX BANACH SPACES

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(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. We consider the problem of the existence and uniqueness of fixed points in X of holomorphic maps $F: X \to X$ of bounded open convex sets X in complex Banach spaces E. As a result of the Earle–Hamilton theorem, the problem in the case where $F(X)$ lies strictly inside X (i.e., $\operatorname{dist}[F(X), E\backslash X] > 0$) has a solution. In this article we show that this problem is also solved in the case where $F(X)$ does not lie strictly inside X (i.e., $\operatorname{dist}[F(X), E\backslash X] = 0$) whenever: (i) F is compact; (ii) F is continuous on $\overline{X}$ and $F(\overline{X}) \subset \overline{X}$; (iii) F has no fixed points on ∂X; and (iv) for each $x \in X$, 1 is not contained in the spectrum of $DF(x)$.

1. INTRODUCTION

Let X be a bounded open convex subset of a complex Banach space E. A subset X' of X lies strictly inside X if there is an $\varepsilon > 0$ such that $\|x-y\| \geq \varepsilon$ whenever $x \in X'$ and $y \notin X$.

The old papers of Wolff [30] and Denjoy [4] where X is the open unit disk in $\mathbb{C}$ (see also Burckel [1]); and recent papers of Kubota [15], MacCluer [18], and Chen [2] where X is the open unit ball in $\mathbb{C}^n$; Goebel and Reich [7], Hayden and Suffridge [9], Kuczumow and Stachura [16], Stachura [23], Suffridge [24], and Vigué [26] where X is the unit Hilbert ball or the Cartesian product of n unit Hilbert balls; Ky Fan [6] for analytic maps of operators in the sense of functional calculus; and the author [28; 29] where X is a homogeneous domain in a J^*-algebra deal with a few questions concerning the existence of fixed points of holomorphic maps $F: X \to X$ such that $F(X)$ does not lie strictly inside X. In these papers the problem of the uniquenesss of fixed points is not studied. This problem is entirely solved in the case where $F(X)$ lies strictly inside X by Earle and Hamilton [5]:

Theorem 1.1. *Let X be a bounded open connected subset of a complex Banach space E. If a holomorpic map $F: X \to X$ is such that $F(X)$ lies strictly inside X, then F has a unique fixed point in X.*

Received by the editors January 25, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46G20, 47H10.
Key words and phrases. Complex Banach spaces, holomorphic maps, bounded open convex sets, fixed points.

If E is infinite dimensional, then the above condition on $F(X)$ is much weaker than the usual condition of the compact closure in X. When $E = \mathbb{C}^n$ the two conditions coincide and Theorem 1.1 reduces to Hervé [11, p. 83] and Reiffen [20, p. 322].

This remarkable result is the kernel of the study of the fixed point theory of holomorphic maps. It was extended, applied, and studied by Harris [10, Appendix, pp. 102–104; 8], Hayden and Suffridge [10], Hervé [12], Vesentini [25], and others.

The purpose of his paper is to give a condition that guarantees the existence and uniqueness of fixed points of holomorphic maps $F: X \to X$ of bounded open convex subsets X of arbitrary complex Banach spaces E in the case where $F(X)$ does not lie strictly inside X; i.e., when $\operatorname{dist}[F(X), E\backslash X] = 0$. In the proof we use [27], properties of holomorphic maps, and degree theory in complex Banach spaces.

2. Result

To state our result, we need some conventions.

A map F is called holomorphic if the Fréchet derivative of F at x (denoted by $DF(x)$) exists as a bounded complex linear map for each x in the domain of definition of F; see Hille and Phillips [13, Chapters 3, 26] for details.

Let E be an infinite-dimensional complex Banach space. Denote the set of bounded linear operators on E by $B(E)$. Let $\sigma(A)$ denote the spectrum of an element A of $B(E)$.

For a set X, $\overline{X}$ and ∂X stand for the closure of X and the boundary of X, respectively. A subset X' of X lies strictly inside X if there esists $\varepsilon > 0$ such that $x + y \in X$ for all $x \in X'$ and $||y|| < \varepsilon$.

A map $F: X \to X$ is said to be compact if $F(X)$ is contained in a compact subset of E.

Our theorem is as follows:

Theorem 2.1. *Suppose $X \subset E$ is a bounded open convex set and $F: X \to X$ a holomorphic map in X such that $F(X)$ does not lie strictly inside X. If* (i) *F is compact;* (ii) *F is continuous on $\overline{X}$ and $F(\overline{X}) \subset \overline{X}$;* (iii) *$F$ has no fixed points on ∂X; and* (iv) *for each $x \in X$, 1 is not contained in $\sigma(DF(x))$, then F has a unique fixed point in X.*

3. Proof of Theorem 2.1

Write $G = I - F$, I being the identity on E. If Ω denotes the set of all zeros of G in X, then Ω is nonempty by [27] and (iii). It is sufficient to show that Ω contains one element.

We first show that Ω is finite. Towards a contradiction, let x, $x_m \in \Omega$, $m = 1, 2, \ldots$, be such that $||x_m - x|| \to 0$ as $m \to \infty$; by the compactness of F, the results of [27], and assumptions (i)–(iii), such a point x and a sequence (x_m) exist. Now since the map $u \to DG(u)$ is a holomorphic map of X into

$\mathscr{L}(E, E)$ (see Nachbin [19, Proposition 3, p. 29]), by the inverse map theorem for complex Banach spaces (see Rudin [21, p. 243]), the map G is locally biholomorphic in X; i.e., in particular, there exists a neighbourhood U of x in X that $G(U) = V$ is open in E, G^{-1} exists, and G^{-1} is holomorphic in V. If m is sufficiently large, however, $x_m \in U$. This yields a contradiction.

If $\Omega = \{x_1, x_2, \dots, x_n\}$ then by Schwartz [22],

$$\deg(G, X, 0) = \deg(G, X\backslash D, 0) = \sum_{k=1}^{n} \deg(G, D_k, 0), \tag{3.1}$$

where D_k are small neighbourhoods of x_k such that the sets $\overline{D}_k$ are pairwise disjoint, $\overline{D}_k \subset X$, and $D = \overline{X} \backslash \left(\bigcup_{k=1}^{n} D_k\right)$.

Without loss of generality we assume that $0 \in X$. (Otherwise we replace the map F and the set X by the map F_0 and the set $X - a$, where F_0: $X - a \to X - a$ is defined by $F_0(u) = F(u + a) - a$, $a \in X$ is arbitrary and fixed.)

Since the implications $tF(\overline{X}) \subset \overline{X}$ hold for $0 \le t \le 1$, we clearly have

$$\{0(1-t) + tF(u)\} \cap \partial X = \varnothing \quad \text{for } u \in \partial X \text{ and } 0 \le t < 1.$$

With (iii) this leads to

$$\{0\} \cap h_t(\partial x) = \varnothing \quad \text{for all } 0 \le t \le 1,$$

where

$$h_t(u) = u - tF(u), \qquad u \in \overline{X},\ 0 \le t \le 1.$$

Applying the homotopy property to h_t (see Cronin [3], Lloyd [17], Schwartz [22]), we get

$$\deg(G, X, 0) = \deg(I, X, 0) = 1. \tag{3.2}$$

Further, since E is complex, the multiplicity of each eigenvalue of $DF(x_k)$, $k = 1, 2, \dots, n$, is even. By Krasnoselskii [14, Theorem 4.7] this gives

$$\deg(G, D_k, 0) = 1, \qquad k = 1, 2, \dots, n. \tag{3.3}$$

From (3.1)–(3.3) we deduce that $n = 1$. This completes the proof.

4. Examples

Let $E = \mathbb{C}$ and $X = \{x \in \mathbb{C}: |x| < 1\}$.

(a) If F is not assumed to satisfy (iii) or (iv), the result of Theorem 2.1 need not hold. In fact, we have constructed a simple example of F,

$$F(x) = (x^2 + 1)/2, \qquad x \in X,$$

that maps X into X, where $F(X)$ does not lie strictly inside X, and F is nonexpansive and compact in X, holomorphic in some neighbourhood of X, but has no fixed points in X.

(b) Let

$$F(x) = -(x - c)/(1 - \bar{c}x), \qquad 0 < |c| < 1,\ x \in X.$$

If $x \in X$ is arbitrary and fixed, then $DF(x)h - h = 0$ whenever $h = 0$. Thus F satisfies the assumptions of Theorem 2.1. We have

$$\operatorname{Fix} F = \{c[1 + (1 - |c|^2)^{1/2}]^{-1}\} \subset X, \qquad 0 < |c| < 1.$$

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