ON THE EXISTENCE AND UNIQUENESS OF FIXED POINTS FOR HOLOMORPHIC MAPS IN COMPLEX BANACH SPACES

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Abstract. We consider the problem of the existence and uniqueness of fixed points in $X$ of holomorphic maps $F: X \rightarrow X$ of bounded open convex sets $X$ in complex Banach spaces $E$. As a result of the Earle–Hamilton theorem, the problem in the case where $F(X)$ lies strictly inside $X$ (i.e., $\text{dist}(F(X), E \setminus X) > 0$) has a solution. In this article we show that this problem is also solved in the case where $F(X)$ does not lie strictly inside $X$ (i.e., $\text{dist}(F(X), E \setminus X) = 0$) whenever: (i) $F$ is compact; (ii) $F$ is continuous on $X$ and $F(X) \subset X$; (iii) $F$ has no fixed points on $\partial X$; and (iv) for each $x \in X$, 1 is not contained in the spectrum of $DF(x)$.

1. Introduction

Let $X$ be a bounded open convex subset of a complex Banach space $E$. A subset $X'$ of $X$ lies strictly inside $X$ if there is an $\varepsilon > 0$ such that $\|x - y\| \geq \varepsilon$ whenever $x \in X'$ and $y \notin X$.

The old papers of Wolff [30] and Denjoy [4] where $X$ is the open unit disk in $\mathbb{C}$ (see also Burckel [1]); and recent papers of Kubota [15], MacCluer [18], and Chen [2] where $X$ is the open unit ball in $\mathbb{C}^n$; Goebel and Reich [7], Hayden and Suffridge [9], Kuczumow and Stachura [16], Stachura [23], Suffridge [24], and Vigué [26] where $X$ is the unit Hilbert ball or the Cartesian product of $n$ unit Hilbert balls; Ky Fan [6] for analytic maps of operators in the sense of functional calculus; and the author [28; 29] where $X$ is a homogeneous domain in a $J^*$-algebra deal with a few questions concerning the existence of fixed points of holomorphic maps $F: X \rightarrow X$ such that $F(X)$ does not lie strictly inside $X$. In these papers the problem of the uniqueness of fixed points is not studied. This problem is entirely solved in the case where $F(X)$ lies strictly inside $X$ by Earle and Hamilton [5]:

Theorem 1.1. Let $X$ be a bounded open connected subset of a complex Banach space $E$. If a holomorphic map $F: X \rightarrow X$ is such that $F(X)$ lies strictly inside $X$, then $F$ has a unique fixed point in $X$.
If $E$ is infinite dimensional, then the above condition on $F(X)$ is much weaker than the usual condition of the compact closure in $X$. When $E = \mathbb{C}^n$ the two conditions coincide and Theorem 1.1 reduces to Hervé [11, p. 83] and Reiffen [20, p. 322].

This remarkable result is the kernel of the study of the fixed point theory of holomorphic maps. It was extended, applied, and studied by Harris [10, Appendix, pp. 102-104; 8], Hayden and Suffridge [10], Hervé [12], Vesentini [25], and others.

The purpose of his paper is to give a condition that guarantees the existence and uniqueness of fixed points of holomorphic maps $F : X \to X$ of bounded open convex subsets $X$ of arbitrary complex Banach spaces $E$ in the case where $F(X)$ does not lie strictly inside $X$; i.e., when $\text{dist}(F(X), E \setminus X) = 0$. In the proof we use [27], properties of holomorphic maps, and degree theory in complex Banach spaces.

### 2. Result

To state our result, we need some conventions.

A map $F$ is called holomorphic if the Fréchet derivative of $F$ at $x$ (denoted by $DF(x)$) exists as a bounded complex linear map for each $x$ in the domain of definition of $F$; see Hille and Phillips [13, Chapters 3, 26] for details.

Let $E$ be an infinite-dimensional complex Banach space. Denote the set of bounded linear operators on $E$ by $B(E)$. Let $\sigma(A)$ denote the spectrum of an element $A$ of $B(E)$.

For a set $X$, $\overline{X}$ and $\partial X$ stand for the closure of $X$ and the boundary of $X$, respectively. A subset $X'$ of $X$ lies strictly inside $X$ if there exists $\varepsilon > 0$ such that $x + y \in X$ for all $x \in X'$ and $||y|| < \varepsilon$.

A map $F : X \to X$ is said to be compact if $F(X)$ is contained in a compact subset of $E$.

Our theorem is as follows:

**Theorem 2.1.** Suppose $X \subset E$ is a bounded open convex set and $F : X \to X$ a holomorphic map in $X$ such that $F(X)$ does not lie strictly inside $X$. If (i) $F$ is compact; (ii) $F$ is continuous on $X$ and $F(X) \subset X$; (iii) $F$ has no fixed points on $\partial X$; and (iv) for each $x \in X$, $1$ is not contained in $\sigma(DF(x))$, then $F$ has a unique fixed point in $X$.

### 3. Proof of Theorem 2.1

Write $G = I - F$, $I$ being the identity on $E$. If $\Omega$ denotes the set of all zeros of $G$ in $X$, then $\Omega$ is nonempty by [27] and (iii). It is sufficient to show that $\Omega$ contains one element.

We first show that $\Omega$ is finite. Towards a contradiction, let $x, x_m \in \Omega$, $m = 1, 2, \ldots$, be such that $||x_m - x|| \to 0$ as $m \to \infty$; by the compactness of $F$, the results of [27], and assumptions (i)-(iii), such a point $x$ and a sequence $(x_m)$ exist. Now since the map $u \to DG(u)$ is a holomorphic map of $X$ into
\( \mathcal{L}(E, E) \) (see Nachbin [19, Proposition 3, p. 29]), by the inverse map theorem for complex Banach spaces (see Rudin [21, p. 243]), the map \( G \) is locally biholomorphic in \( X \); i.e., in particular, there exists a neighbourhood \( U \) of \( x \) in \( X \) that \( G(U) = V \) is open in \( E \), \( G^{-1} \) exists, and \( G^{-1} \) is holomorphic in \( V \). If \( m \) is sufficiently large, however, \( x_m \in U \). This yields a contradiction.

If \( \Omega = \{x_1, x_2, \ldots, x_n\} \) then by Schwartz [22],

\[
\tag{3.1} \deg(G, X, 0) = \deg(G, X \setminus D, 0) = \sum_{k=1}^{n} \deg(G, D_k, 0),
\]

where \( D_k \) are small neighbourhoods of \( x_k \) such that the sets \( \overline{D_k} \) are pairwise disjoint, \( \overline{D_k} \subset X \), and \( D = \overline{X \setminus \bigcup_{k=1}^{n} D_k} \).

Without loss of generality we assume that \( 0 \in X \). (Otherwise we replace the map \( F \) and the set \( X \) by the map \( F_0 \) and the set \( X - a \), where \( F_0 : X - a \to X - a \) is defined by \( F_0(u) = F(u + a) - a \), \( a \in X \) is arbitrary and fixed.)

Since the implications \( tF(X) \subset X \) hold for \( 0 < t < 1 \), we clearly have

\[
\{0(1 - t) + tF(u)\} \cap \partial X = \emptyset \quad \text{for } u \in \partial X \text{ and } 0 \leq t < 1.
\]

With (iii) this leads to

\[
\{0\} \cap h_t(\partial X) = \emptyset \quad \text{for all } 0 \leq t \leq 1,
\]

where

\[
h_t(u) = u - tF(u), \quad u \in \overline{X}, \quad 0 \leq t \leq 1.
\]

Applying the homotopy property to \( h_t \) (see Cronin [3], Lloyd [17], Schwartz [22]), we get

\[
\tag{3.2} \deg(G, X, 0) = \deg(I, X, 0) = 1.
\]

Further, since \( E \) is complex, the multiplicity of each eigenvalue of \( DF(x_k) \), \( k = 1, 2, \ldots, n \), is even. By Krasnoselskii [14, Theorem 4.7] this gives

\[
\tag{3.3} \deg(G, D_k, 0) = 1, \quad k = 1, 2, \ldots, n.
\]

From (3.1)—(3.3) we deduce that \( n = 1 \). This completes the proof.

4. Examples

Let \( E = \mathbb{C} \) and \( X = \{x \in \mathbb{C} : |x| < 1\} \).

(a) If \( F \) is not assumed to satisfy (iii) or (iv), the result of Theorem 2.1 need not hold. In fact, we have constructed a simple example of \( F \),

\[
F(x) = (x^2 + 1)/2, \quad x \in X,
\]

that maps \( X \) into \( X \), where \( F(X) \) does not lie strictly inside \( X \), and \( F \) is nonexpansive and compact in \( X \), holomorphic in some neighbourhood of \( X \), but has no fixed points in \( X \).

(b) Let

\[
F(x) = -(x - c)/(1 - \overline{c}x), \quad 0 < |c| < 1, \quad x \in X.
\]
If $x \in X$ is arbitrary and fixed, then $DF(x)h - h = 0$ whenever $h = 0$. Thus $F$ satisfies the assumptions of Theorem 2.1. We have

$$\text{Fix } F = \{c[1 + (1 - |c|^2)^{1/2}]^{-1} \} \subset X, \quad 0 < |c| < 1.$$ 

References


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