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MEAGER-NOWHERE DENSE GAMES (II): CODING STRATEGIES

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ABSTRACT. We consider three closely related infinite two person games in which the second player has a winning perfect information strategy and examine under what conditions it suffices for the second player to remember only the most recent two moves in the game in order to insure a win. Strategies depending on this information only are called coding strategies.

Let (X, τ) be a T_1-space, and let J be its ideal of nowhere dense sets. $\langle J\rangle$ denotes the collection of meager (also known as first category-) subsets of X. Throughout the paper we assume that the T_1-spaces under consideration do not have isolated points (the assumption is not important and only made to avoid trivialites). Consider the following game of length ω, denoted by $\mathrm{RG}(J)$. In the first inning ONE picks a meager set B_1, and TWO responds with a nowhere dense set W_1. In the second inning ONE picks a meager set B_2 (which does not have to bear any relation to sets picked in previous innings), and TWO responds with a nowhere dense set W_2, and so on. The players play an inning for each positive integer and construct a sequence

$$(B_1, W_1, \ldots, B_m, W_m, \ldots),$$

where for each positive integer m, B_m is ONE's meager set picked during inning m and W_m is TWO's nowhere dense set picked during the inning. Such a sequence is called a play of $\mathrm{RG}(J)$, and TWO is declared the winner of this play if $\bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} W_n$. It is an elementary exercise to show that TWO has a winning perfect information strategy in the game $\mathrm{RG}(J)$, i.e. there is a function $F\colon {}^{<\omega}\langle J\rangle \to J$ such that every play

$$(B_1, W_1, \ldots, B_m, W_m, \ldots)$$

of $\mathrm{RG}(J)$ with $W_m = F(B_1, \ldots, B_m)$ for each positive integer m is won by TWO. We say that TWO followed strategy F and call the resulting play an F-play.

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In this paper we investigate if TWO really needs this much memory to insure a win. A function $F\colon J \times \langle J\rangle \to J$ is called a coding strategy for TWO and will be called a winning coding strategy for TWO in the game $\mathrm{RG}(J)$ if every play $(B_1, W_1, \ldots, B_m, W_m, \ldots)$ of $\mathrm{RG}(J)$ with $W_1 = F(\emptyset, B_1)$ and $W_{n+1} = F(W_n, B_{n+1})$ for each positive integer n is won by TWO. Intuitively speaking, a coding strategy is a strategy that requires remembering only the most recent move of TWO and of ONE to decide what move to make next. The choice of the term "coding strategy" is motivated by the methods used to show that a player has a winning strategy of this kind.

In the first section we characterize those T_1-spaces for which TWO has a winning coding stategy in the game $\mathrm{RG}(J)$ (Theorem 2). In the second section we consider coding strategies for TWO in the game $\mathrm{WMG}(J)$, which is just $\mathrm{RG}(J)$ with the additional rule that ONE must also play $B_m \subseteq B_{m+1}$ for each positive integer m. We show that if the generalized continuum hypothesis is true then TWO has a winning coding strategy in the game $\mathrm{WMG}(J)$ for all T_1-spaces (Theorem 5). We suspect that the additional set theoretic hypothesis used in the proof is superfluous and that this result is simply a theorem of ZFC (Zermelo–Fraenkel set theory, including the axiom of choice). In the third and final section of this paper we consider the existence of winning coding strategies of TWO in the game $\mathrm{WMEG}(J)$, which is just the game $\mathrm{WMG}(J)$ with the additional rule that TWO wins only if $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} W_n$. It is not clear for which T_1-spaces TWO has a winning coding strategy in the game $\mathrm{WMEG}(J)$. We merely give evidence that the answer is not simply "all T_1-spaces" (Theorem 6) or "no T_1-spaces" (Theorem 7).

Our terminology and notation is standard and can be found in most current textbooks on set theory. [2] or [6] should be sufficient references for the basic set theoretic facts that we use. Two other versions of the games considered here were introduced in [3] where we were interested in a different type of winning strategy for TWO. The reader interested in the original motivation for considering meager-nowhere dense games could consult the introductory discussion of [3]. Readers interested in reading more about topological games should read Telgarsky's survey article [5].

We thank Professor Fred Galvin for patiently listening to some proofs and for his very useful commentary on the matters we discuss here.

1. The random game on J, $\mathrm{RG}(J)$

Recall that a family J of subsets of an infinite set S is called a free ideal on S if: S is not in J, every finite subset of S is in J; if A is in J and B is a subset of A, then B is in J; and if A and B are in J, so is $A \cup B$. For a free ideal J on S we denote by $\langle J\rangle$ the smallest family of subsets of S that contains J and is closed under countable unions; we call $\langle J\rangle$ the σ-completion of J. The following statements are equivalent for a family J of subsets of S:

(a) J is a free ideal on S.

(b) There is a T_1-topology τ on S such that (i) (S, τ) has no isolated points, and (ii) J is the collection of nowhere dense subsets of S. (Proof of (a) $\Rightarrow$ (b). Put a set in τ if it is the complement of a set in J. Then τ is a topology on S, which is T_1 because J is free; τ satisfies (i) because S is not in J, and (ii) because any set not in J is dense.) In view of this equivalence we henceforth discuss matters using the more convenient notion of a free ideal and its σ-completion instead of the equivalent topological terminology.

The following fact is the main device used to construct coding strategies for TWO in the games considered in this paper.

Lemma 1 (The Coding Lemma). *Let $(P, <)$ be a partially ordered set such that for every p in P, $|\{q \in P: p < q\}| = |P|$. For every family of sets H with $|H| \leq |P|$ there is a function Φ: $P \to {}^{<\omega}H$ such that for each p in P and each $(h_1, \dots, h_n) \in {}^{<\omega}H$ there is a $q \in P$ with $p < q$ and $\Phi(q) = (h_1, \dots, h_n)$.*

Proof. By the hypotheses P is infinite. Let ${}^{<\omega}H = \{t_\xi: \xi < |P|\}$ be a surjective enumeration of ${}^{<\omega}H$. For p in P put $A_p = \{q \in P: p \leq q\}$. Then $\{A_p: p$ in $P\}$ is a family of $|P|$ many sets, each of size $|P|$. Choose (see [6, p. 53, Lemma 2.7.2]) a pairwise disjoint family $\{Q_\xi: \xi < |P|\}$ for which for all $\xi < |P|$,

(i) $|Q_\xi| = |P|$,
(ii) $Q_\xi \subset P$, and
(iii) $Q_\xi \cap A_p \neq \varnothing$ for each p in P.

Define Φ: $P \to {}^{<\omega}H: p \to \Phi(p)$ by $\Phi(p) = t_\xi$ if $p \in Q_\xi$, and $\Phi(p) = t_0$ otherwise. Then Φ has the required properties $\square$

We also use the following notation in our exposition. For $J \subset \wp(S)$ a free ideal and X a subset of S, J_X denotes the collection of sets in J that are subsets of X. We call J_X the relativization of J to X.

Theorem 2. *For a free ideal $J \subset \wp(S)$ the following statements are equivalent.*

(a) TWO *has a winning coding strategy in* $\mathrm{RG}(J)$.
(b) $\mathrm{cof}(\langle J\rangle, \subset) \leq |J|$.

Proof. (a) $\Rightarrow$ (b). Let $F: J \times \langle J\rangle \to J$ be a winning coding strategy for TWO in $\mathrm{RG}(J)$. Define f: $J \to \langle J\rangle$ as follows. Let $W \in J$ and put

$$W_0 = W \quad \text{and} \quad W_{n+1} = F(W_n, \varnothing) \quad \text{for all } n \in \omega.$$

Define $f(W) = \bigcup_{n=0}^{\infty} W_n$. Then $\{f(W): W \in J\} \subset \langle J\rangle$ is cofinal in $\langle J\rangle$. For let $B \in \langle J\rangle$ and let $W = F(\varnothing, B)$. Since $(B, W_0, \varnothing, W_1, \varnothing, W_2, \dots, \varnothing, W_n, \dots)$ is an F-play of $\mathrm{RG}(J)$, it is won by TWO, hence $B \subseteq f(W)$.

(b) $\Rightarrow$ (a). Since (a) is trivially true when S is in $\langle J\rangle$ we assume that $S \notin \langle J\rangle$. Choose an X in J for which $J(X) = \{Y \in J: X \subset Y\}$ has minimal cardinality.

Case 1. $|J| = |J(X)|$. Then $(J(X), \subset)$ is a partially ordered set having the property that for each Y in $J(X)$, $|\{Z \in J(X): Y \subseteq Z\}| = |J(X)|$. Choose $H \subseteq \langle J\rangle$ cofinal of minimal size. Choose a function f: $\langle J\rangle \to H$ such that for each B in $\langle J\rangle$, $B \subseteq f(B)$. By the coding lemma choose a function Φ: $J(X) \to {}^{<\omega}H$ with the property that for each $(D_1, \dots, D_n)$ in ${}^{<\omega}H$ and each U in $J(X)$, there is a V in $J(X)$ with $U \subseteq V$ and $\Phi(V) = (D_1, \dots, D_n)$. Let Ψ be a winning perfect information strategy for TWO in $\mathrm{RG}(J)$. Define a coding strategy $F: J \times \langle J\rangle \to J$ for TWO as follows: Let (W, B) be in $J \times \langle J\rangle$.

Possibility 1. $W = \varnothing$. Choose U in $J(X)$ with $U \supseteq \Psi(f(B))$ and $\Phi(U) = (f(B))$. Put $F(W, B) = U$.

Possibility 2. $\varnothing \neq W$ is in $J(X)$ and $\Phi(W) = (D_1, \dots, D_n)$. Choose U in $J(X)$ with $U \supseteq \Psi(D_1, \dots, D_n, f(B))$ and $\Phi(U) = (D_1, \dots, D_n, f(B))$. Put $F(W, B) = U$.

Possibility 3. In all other cases put $F(W, B) = \varnothing$.

An inductive computation shows that if $(B_1, W_1, \dots, B_n, W_n, \dots)$ is an F-play of $\mathrm{RG}(J)$, then W_1 is defined by Possibility 1, and W_n for $n > 1$ is defined by Possibility 2, and that this play is won by TWO.

Case 2. $|J(X)| < |J|$. Then $|\wp(X)| = |J|$ (and so X is infinite), for clearly $|J| = |\wp(X)| + |J(X)|$. Fix $H \subseteq \langle J\rangle \backslash J$ cofinal of minimal size such that for each $Z \in H$, $X \subset Z$. Next write $X = \bigcup_{n=1}^{\infty} X_n$ where $\{X_n : n \text{ a positive integer}\}$ is pairwise disjoint and for each positive integer n, $|X_n| = |X|$. For each positive integer n choose an injection $\Phi_n| : {}^{<\omega}H \to \wp(X_n)\backslash\{X_n, \varnothing\}$. Choose a function $f: \langle J\rangle \to H$ such that for each B in $\langle J\rangle B \subseteq f(B)$ and for each B in H write $B = g(B) \cup X$ with $X \cap g(B) = \varnothing$. Let Ψ be a winning perfect information strategy of TWO in $\mathrm{RG}(J_{S\backslash X})$. Define a coding strategy $F: J \times \langle J\rangle \to J$ for TWO as follows. Let (W, B) in $J \times \langle J\rangle$ be given.

Possibility 1. $W = \varnothing$. Put $F(W, B) = \Psi(g(f(B))) \cup \Phi_1((f(B)))$.

Possibility 2. $W \neq \varnothing$ and for some positive integer n

(i) for $m < n$, $X_m \subset W$;
(ii) $W \cap X_n$ is in $\wp(X_n)\backslash\{X_n, \varnothing\}$;
(iii) for $m > n$, $W \cap X_m = \varnothing$;
(iv) $W \cap X_n$ is in the range of Φ_n; and
(v) $\Phi_n^{-1}(W \cap X_n) = (D_1, \dots, D_n)$.

Put

$$F(W, B) = \Psi(g(D_1), \dots, g(D_n),$$
$$g(f(B))) \cup \left(\bigcup_{j=1}^{n} X_j\right) \cup \Phi_{n+1}(D_1, \dots, D_n, f(B)).$$

Possibility 3. In all other cases put $F(W, B) = \varnothing$.

An inductive computation shows that if $(B_1, W_1, \dots, B_n, W_n, \dots)$ is an F-play of $\mathrm{RG}(J)$, then W_1 is defined by Possibility 1, and W_n for $n > 1$ is defined by Possibility 2, and that this play is won by TWO. This completes the proof of the theorem. □

Example 1. $J = \{N \subset \mathbb{R}: N$ is nowhere dense$\}$ ($\mathbb{R}$ is the real line).

In this case: $\mathrm{cof}(\langle J\rangle, \subset) \leq 2^{\aleph_0} < |J|$ and so by the theorem TWO has a winning coding strategy in $\mathrm{RG}(J)$.

Example 2. $J = [\kappa]^{<\lambda}$ where $\omega = \mathrm{cof}(\lambda) \leq \lambda < \kappa$ are infinite cardinals.

Then $\langle J\rangle = [\kappa]^{\leq\lambda}$. By the theorem the following are equivalent:

(a) TWO has a winning coding strategy in $\mathrm{RG}(J)$.

(b) $\mathrm{cof}([\kappa]^{\leq\lambda}, \subset) \leq \kappa^{<\lambda}$.

Suppose that κ has countable cofinality: if $\lambda = \omega$, then (b) fails (by a classical result of König; see e.g. [2, p. 43, Lemma 6.2 and p. 46, Corollary 4]). Thus it is a theorem of ZFC that each uncountable cardinal number with countable cofinality carries a free ideal J such that player TWO does not have a winning coding strategy in the game $\mathrm{RG}(J)$. The situation for cardinal numbers of uncountable cofinality is more complicated.

Corollary 3 (GCH). *Whenever κ is a cardinal of uncountable cofinality and $J \subset \wp(\kappa)$ is a free ideal, then TWO has a winning coding strategy in* $\mathrm{RG}(J)$.

Proof. Note that $\kappa \leq |J| \leq \kappa^+$. But now $|\langle J\rangle| \leq |J|^{\aleph_0}$, which is κ if $|J| = \kappa$ by GCH as $\mathrm{cof}(\kappa)$ is uncountable (see e.g. [2, p. 49, Corollary 2]) and which is κ^+ if $|J| = \kappa^+$, again because GCH holds. Thus $\mathrm{cof}(\langle J\rangle, \subset) \leq |J|$. Apply Theorem 2. □

It is natural to ask if an additional set theoretic hypothesis such as GCH is needed to prove the assertion of Corollary 3. In [4] we investigate this question further.

2. The weakly monotonic game, $\mathrm{WMG}(J)$

Fix a free ideal $J \subset \wp(S)$ and its σ-completion $\langle J\rangle$. Recall that an infinite sequence $(B_1, W_1, \ldots, B_n, W_n, \ldots)$ is a play of $\mathrm{WMG}(J)$ if for each positive integer n, $B_n \subseteq B_{n+1} \in \langle J\rangle$ and $W_n \in J$. TWO wins this play of $\mathrm{WMG}(J)$ if

$$\bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} W_n.$$

Since every play of $\mathrm{WMG}(J)$ is also a play of $\mathrm{RG}(J)$ it follows from Corollary 3 that if GCH is true and if the cardinality of the underlying set S has uncountable cofinality, then TWO has a winning coding strategy in $\mathrm{WMG}(J)$. Lemma 4 enables us to proved this result for the case when the cardinality of S has countable cofinality.

Lemma 4 (Induction from countably many summands). *Let $J \subset \wp(S)$ be a free ideal and assume that for some partition $S = \bigcup_{n=1}^{\infty} S_n$ (where $\{S_n: n$ a positive integer$\}$ is a collection of pairwise disjoint nonempty sets), TWO has a winning coding strategy in $\mathrm{WMG}(J_{S_n})$ for each n. Then TWO has a winning coding strategy in* $\mathrm{WMG}(J)$.

Proof. For each n choose $x_n \in S_n$ and a winning coding strategy F_n for TWO in $\mathrm{WMG}(J_{S_n})$. Define a coding strategy $F\colon J \times \langle J \rangle \to J$ for TWO as follows: Let (W, B) from $J \times \langle J \rangle$ be given.

Case 1. $W = \varnothing$. Put $F(W, B) = F_1(W, B \cap S_1) \cup \{x_2\}$.

Case 2. There is an integer $n > 1$ such that

(i) $W \cap S_n$ contains exactly one point while

(ii) $W \cap S_m = \varnothing$ for $m > n$.

Then n is unique. Let $F(W, B)$ be the set

$$\left(\bigcup_{j=1}^{n-1} F_j(W \cap S_j, B \cap S_j)\right) \cup F_n(\varnothing, B \cap S_n) \cup \{x_{n+1}\}.$$

Case 3. In all other cases put $F(W, B) = \varnothing$.

Since the unions involved in the definition of F are all finite, F certainly is a coding strategy of TWO. We show that F is winning for TWO. Let $(B_1, W_1, \dots, B_n, W_n, \dots)$ be a play of $\mathrm{WMG}(J)$ during which TWO used F. Then $W_1 = F(\varnothing, B_1)$ and $W_{n+1} = F(W_n, B_{n+1})$ for each positive integer n. An inductive computation shows that W_1 is defined by Case 1 and W_n is defined by Case 2 for all integers $n > 1$. The result then follows from the fact that each of the coding stategies F_n is winning for TWO on the particular part of S, and that these are progressively used to build TWO's responses. □

We are now ready to prove Theorem 5.

Theorem 5 (GCH). *Whenever $J \subset \wp(S)$ is a free ideal, TWO has a winning coding strategy in* $\mathrm{WMG}(J)$.

Proof. The proof is by induction on $|S|$.

Case 1. $\mathrm{cof}(|S|) > \aleph_0$. Then the result follows from Corollary 3.

Case 2. $\mathrm{cof}(|S|) = \aleph_0$. The result follows from Lemma 4, using the induction hypothesis. □

I don't know a proof of this theorem in ZFC; one certainly does not need the full strength of GCH to derive the conclusion of this theorem. In all the specific examples that we have considered thus far, it turned out that TWO has a winning coding strategy and that this could be proven in ZFC. In particular, if λ is a cardinal number of countable cofinality and S is an infinite set, then TWO has a winning coding strategy in $\mathrm{WMG}(J)$ where J is the collection $[S]^{<\lambda}$. In the case when λ is ω, even more can be said about the coding stategies of TWO. This last observation will be discussed elsewhere in the context of results by Ciesielski and Laver (see [1]) about a game of D. Gale.

We also have a proof in ZFC that if the cardinality of the underlying set S is less than $\aleph_{\omega_1}$ and if $J \subset \wp(S)$ is a free ideal, then TWO has a winning coding strategy in $\mathrm{WMG}(J)$. This evidence leads us to the following conjecture.

Conjecture. *It is a theorem of ZFC that for every infinite set S, whenever $J \subset \wp(S)$ is a free ideal then* TWO *has a winning coding strategy in* $\mathrm{WMG}(J)$.

At this stage the following is the simplest unverified case of this conjecture.

Problem 1. It is a theorem of ZFC that for every free ideal $J \subset \wp(\aleph_{\omega_1})$ TWO has a winning coding strategy in $\mathrm{WMG}(J)$?

3. The weakly monotonic equal game $\mathrm{WMEG}(J)$

Fix a free ideal $J \subset \wp(S)$ and its σ-completion $\langle J \rangle$. A play of $\mathrm{WMEG}(J)$ is an infinite sequence $(B_1, W_1, \ldots, B_n, W_n, \ldots)$ where for each positive integer n, $B_n \subseteq B_{n+1} \in \langle J \rangle$ and $W_n \in J$. TWO wins this play of $\mathrm{WMEG}(J)$ if $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} W_n$. This section is a discussion of how serious a restriction on TWO this winning condition (as opposed to that of $\mathrm{WMG}(J)$) is. We have not yet characterized those free ideals J for which TWO has a winning coding strategy in $\mathrm{WMEG}(J)$. In this section we merely give evidence that the answer is not simply "All free ideals" or "No free ideals."

Theorem 6. *Let κ be an uncountable cardinal number and let $J = [\kappa]^{<\aleph_0}$. Then TWO does not have a winning coding strategy in $\mathrm{WMEG}(J)$; in fact, TWO does not even have a winning strategy of the form*

$$W_1 = F(B_1), \qquad W_{n+1} = F(W_1, \ldots, W_n, B_{n+1}), \qquad n \text{ a positive integer},$$

in $\mathrm{WMEG}(J)$.

Proof. It suffices to prove this theorem for $\kappa = \omega_1$ and for strategies F as above, which also have the property that $F(B) \subseteq B$ and $F(W_1, \ldots, W_n, B) \subseteq B$ for $W_1, \ldots, W_n \in J$ and $B \in \langle J \rangle$. $(**)$

Put $\Lambda = \{\gamma < \omega_1 : \gamma \text{ is a limit ordinal}\}$ and let F be such a strategy of TWO.

Claim. There is a sequence $((T_n, S_n, \alpha_n) : n \in \omega)$ such that for all (appropriate) n in ω

(i) $T_n \in [\omega_1]^{<\aleph_0}$ and $\alpha_n = \max(T_n) < S_n \subset \Lambda$;
(ii) $S_n = \{\gamma \in S_{n-1} : F(T_0, \ldots, T_{n-1}, \gamma) = T_n\}$ is stationary in ω_1; and
(iii) if $\gamma_j \in S_j$ and $\gamma_j < \gamma_{j+1}$ for $0 \le j < n$ then $(\gamma_0, T_0, \ldots, \gamma_n, T_n)$ is an F-position in $\mathrm{WMEG}(J)$.

Proof of the claim. Such a sequence is constructed by induction.

Step 1. Define $f : \Lambda \to \omega_1 : \gamma \to f(\gamma) = \max F(\gamma) (< \gamma)$. Since f is regressive and Λ stationary by Fodor's lemma (see e.g. [6, p. 11, Lemma 1.2.12]) pick an $\alpha_0 \in \omega_1$ and a stationary subset R_0 of ω_1 with f constant of value α_0 on R_0. $R_0 = \bigcup \{\gamma \in R_0 : F(\gamma) = T \text{ and } T \in [\alpha_0]^{<\aleph_0}\}$ is a countable partition whence for some T_0 in $[\alpha_0]^{<\aleph_0}$, $S_0 = \{\gamma \in R_0 : F(\gamma) = T_0\}$ is a stationary subset of ω_1. This specifies (T_0, S_0, α_0).

Step $n+1$. Now suppose that $((T_j, S_j, \alpha_j) : j \in n+1)$ has been constructed satisfying (i)–(iii). Define $f : S_n \to \omega_1 : \gamma \to f(\gamma) = \max F(T_0, \ldots, T_n, \gamma)$. Then f is regressive and by (ii) S_n is stationary in ω_1. By Fodor's lemma pick an $\alpha_{n+1} \in \omega_1$ and a stationary subset, R_{n+1}, of ω_1 with f constant of value α_{n+1} on R_{n+1}. $R_{n+1} = \bigcup\{\gamma \in R_{n+1} : F(\gamma) = T$ and $T \in [\alpha_{n+1}]^{<\aleph_0}\}$ is a countable partition so for some T_{n+1} in $[\alpha_{n+1}]^{<\aleph_0}$, $S_{n+1} = \{\gamma \in R_{n+1} : F(T_0, \dots, T_n, \gamma) = T_{n=1}\}$ is stationary in ω_1. This specifies $((T_j, S_j, \alpha_j) : j \in n+2)$ having properties (i), (ii), and (iii). Thus the claim follows by induction.

Fix a sequence $((T_n, S_n, \alpha_n) : n \in \omega)$ as in the claim and let $\alpha = \sup\{\alpha_n : n \in \omega\} + 1$. For each $n \in \omega$ pick a $B_n \in S_n$ with $\alpha < \sup B_n < \sup B_{n+1}$. Then $(B_0, T_0, \dots, B_n, T_n, \dots)$ is an F-play of WMEG(J), which is lost by TWO. This completes the proof of the proposition. □

On the other hand we have:

Theorem 7. *Let $J \subset \wp(S)$ be a free ideal on a set S and assume that*

(i) $\{2^\kappa : \aleph_0 \leq \kappa \leq |S|\}$ *is a finite set;*
(ii) *for each Y in $\langle J \rangle \backslash J$ there is a subset Z of Y with Z in J and $2^{|Z|} = 2^{|Y|}$; and*
(iii) *for each Y in $\langle J \rangle \backslash J$, $|\wp(Y)| = |[S]^{\leq |Y|}|$.*

Then TWO has a winning coding strategy in WMEG(J).

Proof. Fix a well-ordering $\ll$ of $\{N \in J : |N| \geq \aleph_0\}$. For each Y in $\langle J \rangle \backslash J$ let $f(Y)$ be the $\ll$-minimum element of J with $2^{|f(Y)|} = 2^{|Y|}$ and $f(Y) \subseteq Y$, and write $f(Y) = \bigcup_{n=1}^{\infty} f_n(Y)$ where $\{f_n(Y) : n$ a positive integer$\}$ is a pairwise disjoint family with

(i) $|f_n(Y)| = |f(Y)|$ for each n, and
(ii) if $f(Y) = f(Z)$ then $f_n(Y) = f_n(Z)$ for each n.

Let $\Phi_n^{f(Y)} : \wp(f_n(Y)) \backslash \{f_n(Y), \varnothing\} \to {}^{<\omega}([S]^{\leq |Y|})$ be a bijection for each Y in $\langle J \rangle \backslash J$. Also let F be a winning perfect information strategy of TWO in WMEG(J) with the property that whenever $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n \subseteq \cdots$ are in $\langle J \rangle$ then for all positive integers k and m,

(i) $F(B_k, B_{k+1}, \dots, B_{k+m+j}) \subseteq F(B_k, B_{k+1}, \dots, B_{k+m+j+1})$ for all j; and
(ii) $\bigcup_{j=0}^{\infty} F(B_k, B_{k+1}, \dots, B_{k+m+j}) = \bigcup_{n=1}^{\infty} B_n$.

Observation. By the hypothesis on cardinal arithmetic, if $Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_n \subseteq \cdots$ is in $\langle J \rangle \backslash J$ then there is a positive integer k such that for all integers $n \geq k$, $f(Y_n) = f(Y_k)$. Now I define a coding strategy $H : J \times \langle J \rangle \to J$ for TWO. So let (W, B) be given with W in J and B in $\langle J \rangle$.

Case 1. (a) $W = \varnothing$ and B is in J. Then put $H(W, B) = B$.

(b) $W = \varnothing$ and B is not in J. Let $G = F(B)$, $\varnothing \neq K \subset f_1(B)$ with $\Phi_1^{f(B)}(K) = (B)$ and put $H(W, B) = (G \backslash f(B)) \cup (f(B) \backslash f_1(B)) \cup K$.

Case 2. (a) $W \neq \varnothing$ and B is in J. Then put $H(W, B) = B$.

(b) $W \neq \varnothing$ and B is not in J, and for some positive integer n, (i) $\varnothing \neq f(B) \backslash W \subset f_n(B)$ with (ii) $\Phi_n^{f(B)}(f(B) \backslash W) = (C_1, C_2, \dots, C_n)$ where $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_n \subseteq B$. Let $G = F(C_1, C_2, \dots, C_n, B)$ and $\varnothing \neq K \subset f_{n+1}(B)$

such that $\Phi_{n+1}^{f(B)}(K) = (C_1, C_2, \ldots, C_n, B)$. Put $H(W, B) = (G\backslash f(B)) \cup (f(B)\backslash f_{n+1}(B)) \cup K$.

Case 3. In all other cases put $H(W, B) = \emptyset$.

It is left to the reader to check that H is a winning coding strategy for TWO in WMEG (J). □

Example 1 (continued). $J = \{N \subset \mathbb{R}\colon N$ is nowhere dense$\}$. If we assume for example the continuum hypothesis or Martin's axiom, the hypotheses of Theorem 7 are satisfied for this example and it follows that TWO has a winning coding strategy in WMEG (J). Thus, for this example we at least have the consistency of the statement that TWO has a winning coding strategy in WMEG (J).

It is conceivable however, that one could prove this without additional set theoretic hypotheses. In particular,

Problem 2. It is a theorem of ZFC that TWO has a winning coding strategy in the game WMEG (J) where $J = \{N \subset \mathbb{R}\colon N$ is nowhere dense$\}$?

Example 2 (continued). $J = [\kappa]^{<\lambda}$ where $\omega = \mathrm{cof}(\lambda) \leq \lambda < \kappa$ are infinite cardinals. We showed in Theorem 6 that if $\lambda = \omega$ then TWO does not have a winning coding strategy in these examples. However, when λ is uncountable and κ is less than or equal to the continuum and if the hypotheses of Theorem 7 holds (e.g. when Martin's axiom holds), then TWO has a winning coding strategy in WMEG (J). Thus it is consistent that TWO has a winning coding strategy in WMEG (J) for such J.

Problem 3. Is it a theorem of ZFC that TWO has a winning coding strategy in the game WMEG (J) when $J = [\kappa]^{<\lambda}$ where $\omega = \mathrm{cof}(\lambda) < \lambda < \kappa$ are infinite cardinal numbers?

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