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A NOTE ON GRADED ALGEBRAS

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ABSTRACT. It is proved that if an algebra over a field of characteristic $p > 0$ is graded by a finite p-group, then its ideals having nilpotent intersection with the identity component are nilpotent themselves.

In [2] Krempa proved that if A is a prime algebra over a field F of characteristic $p > 0$ and $F \subseteq K = F(a)$ is a field extension with $a^{p^n} \in F$ for some n, then every ideal I of $A \otimes_F K$ satisfying $A \cap I = 0$ is nilpotent. The proof was based on a use of Martindale's rings of quotients. In this note we give an elementary proof of the following more general result.

Theorem. *Let R be a G-graded F-algebra, where G is a finite p-group and F is a field of characteristic $p > 0$. If I is an ideal of R such that $I \cap R_e$ is nilpotent, then I is nilpotent.*

Throughout, rings and algebras are associative with an identity element 1. All groups are denoted multiplicatively and the group identity is denoted by e.

Let G be a group. Recall that a ring R is G-graded if $R = \bigoplus_{g \in G} R_g$ is the direct sum of additive subgroups R_g of R with $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. It follows that R_e is a subring of R containing 1. An ideal I of R is called homogeneous if $I = \bigoplus_{g \in G} I \cap R_g$. The largest homogeneous ideal of R contained in a given ideal J of R is denoted by J_G.

The proof of the theorem is based on the following three lemmas:

Lemma 1 [1]. *If I is a homogeneous ideal of a ring R graded by a finite group and $I \cap R_e$ is nilpotent, then I is nilpotent.*

Lemma 2. *If A is an F-algebra over a field of characteristic $p > 0$, then every ideal I of the group algebra $A[G]$ of a finite p-group G such that $I \cap A = 0$ is nilpotent.*

Proof. Let $\pi \colon A[G] \to A$ be the augmentation homomorphism, i.e. $\pi(\sum a_g g) = \sum a_g$. It is well known that $(\ker \pi)^n = 0$, where $n = |G|$. Hence for every $a_1, \ldots, a_n \in I$, $(a_1 - \pi(a_1)) \cdots (a_n - \pi(a_n)) = 0$. On the other hand, since I

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is an ideal, $(a_1 - \pi(a_1))\cdots(a_n - \pi(a_n)) = i \pm \pi(a_1)\cdots\pi(a_n)$ for some $i \in I$. Thus $\pi(a_1)\cdots\pi(a_n) \in I \cap A = 0$, which means that $(\pi(I))^n = 0$. Now $I \subseteq \pi(I) + \ker\pi$ and $(\pi(I)+\ker\pi)^n \subseteq \ker\pi$, so I is nilpotent.

Lemma 3. *Let $A = \bigoplus_{h\in H} A_h$ be a ring graded by an abelian group H and let $f\colon A \to A[H]$ be the ring homomorphism of A into the group ring $A[H]$ defined by $f(\sum a_h) = \sum a_h h$, where $a_h \in A_h$.*

(i) *If I is an ideal of A, then the ideal J of $A[H]$ generated by $f(I)$ is equal to $\sum_{g\in H} f(I)g$;*

(ii) *If $I_G = 0$, then $J \cap A = 0$.*

Proof. (i). It suffices to prove that $\sum_{g\in G} f(I)g$ is an ideal of $A[H]$, and for this, since H is abelian, it is enough to check that if $a \in A_x$, then $f(I)a \subseteq \sum_{g\in H} f(I)g$ and $af(I) \subseteq \sum_{g\in H} f(I)g$. Observe that $f(I)ax = f(I)f(a) = f(Ia) \subseteq f(I)$. Hence $f(I)a \subseteq f(I)x^{-1}$. Similarly $af(I) \subseteq \sum_{g\in H} f(I)g$.

(ii). Suppose that $a \in J \cap A$. By (i), a is a finite sum $\sum_g a_g g$, with $a_g \in f(I)$ and $g \in G$. Each a_g is equal to a finite sum $\sum_h i_{h,g} h$, where $i_{h,g} \in A_h$ and $\sum_h i_{h,g} \in I$. Now $\sum_g a_g g = \sum_g(\sum_h i_{h,g} h)g = \sum_x(\sum_{hg=x} i_{h,g})x \in A$, so for every $e \neq x \in G$, $\sum_{hg=x} i_{h,g} = 0$. This and the fact that $i_{h,g} \in A_h$ give that if $hg \neq e$, then $i_{h,g} = 0$. Thus $a_g = i_{g^{-1},g} g$. Now $i_{g^{-1},g} \in I_G = 0$, so for all g, $a_g = 0$. This gives $a = 0$.

Proof of the theorem. We proceed by induction on $|G|$. Let $|G| = p$. Passing, if necessary, to the factor algebra R/I_G, one can assume that $I_G = 0$. Now Lemmas 2 and 3 give immediately that I is nilpotent. If $|G| > p$, then G, being a p-group, contains a nontrivial normal subgroup H. Let $T = \bigoplus_{h\in H} R_h$. Obviously T is an H-graded algebra, so the induction assumption applied to T gives that $I \cap T$ is nilpotent. Now R has a natural structure of G/H-graded algebra with the identity component equal to T. Hence the induction assumption gives that I is nilpotent.

Corollary 1 (cf. [2]). *Let A be an algebra over a field F of characteristic $p > 0$ and let $F \subseteq K = F(a)$ be a field extension with $a^{p^n} \in F$ for some n. Then every ideal I of $A \otimes_F K$ such that $A \cap I = 0$ is nilpotent.*

Proof. Without loss of generality one can assume that n is a minimal integer with $a^{p^n} \in F$. Then $(K\colon F) = p^n$ and $R = A \otimes_F K$ can be treated as an algebra graded by the cyclic group $\langle g\rangle$ of order p^n with homogeneous components $R_{g^k} = A \otimes a^k$, $k = 0, 1, \ldots, p^n - 1$. Now the corollary is an immediate consequence of the theorem.

As an immediate consequence of the theorem one also gets

Corollary 2. *If R is that of the theorem and R_e is Jacobson semisimple, then the Jacobson radical of R is equal to the largest ideal of R not intersecting R_e.*

Of course Corollary 2 is valid for many other radicals as well.

Lemma 2 (so also the theorem) does not hold if I is a one-sided ideal.

Example. Let A be the algebra of all 2×2-matrices over a field F of characteristic 2 and let $G = \{e, g\}$ be a group of order 2. Observe that

$$I = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} e + \begin{pmatrix} c & a \\ d & b \end{pmatrix} g \,|\, a, b, c, d \in F \right\}$$

is a left ideal of $A[G]$ such that $I \cap A = 0$. Now

$$\pi(I) = \left\{ \begin{pmatrix} x & x \\ y & y \end{pmatrix} \,|\, x, y \in F \right\},$$

so

$$(\pi(I))^2 = \pi(I).$$

This shows that I is not nilpotent.

References

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