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THE INDEX OF NORMAL FREDHOLM ELEMENTS OF C^*-ALGEBRAS

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ABSTRACT. Examples are given of normal elements of C^*-algebras that are invertible modulo an ideal and have nonzero index, in contrast to the case of Fredholm operators on Hilbert space. It is shown that this phenomenon occurs only along the lines of these examples.

Let T be a bounded operator on a Hilbert space. If the range of T is closed and both T and T^* have a finite dimensional kernel then T is *Fredholm*, and the index of T is $\dim(\ker T) - \dim(\ker T^*)$. If T is normal then $\ker T = \ker T^*$, so a normal Fredholm operator has index 0.

Let us consider a generalization of the notion of Fredholm operator introduced by Atiyah. Let X be a compact Hausdorff space and consider continuous functions $T: X \to B(H)$, where $B(H)$ is the set of bounded linear operators on a separable infinite dimensional Hilbert space with the norm topology. The set of such functions forms a C^*- algebra $C(X) \otimes B(H)$. A function T is *Fredholm* if $T(x)$ is Fredholm for each x. Atiyah [1, Appendix] showed how such an element has an index which is an element of $K^0(X)$. Suppose that T is Fredholm and $T(x)$ is normal for each x. Is the index of T necessarily 0?

There is a generalization of this question that we would like to consider. Let A be a C^*-algebra, $\mathscr{K}$ the algebra of compact operators on a separable infinite dimensional Hilbert space, and $M(A \otimes \mathscr{K})$ the multiplier algebra of the C^*-completion of the algebraic tensor product of A with $\mathscr{K}$ (see Blackadar [2, 12.2.3]). An element x of $M(A \otimes \mathscr{K})$ that is invertible modulo $A \otimes \mathscr{K}$ is called *Fredholm* and has an index in $K_0(A)$. This index can be developed in analogy with the usual Fredholm theory (see Wegge-Olsen [9] for an elegant treatment) but we shall work with the connecting homorphism of K-theory.

Definition. Let $0 \to A \to B \to B/A \to 0$ be a short exact sequence of C^*-algebras. Let q denote the quotient map of B onto B/A. If $x \in B$ is invertible modulo A then *index*(x) is $\partial[q(x)]$, where $\partial: K_1(B/A) \to K_0(A)$ is

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the connecting homomorphism and $[q(x)]$ denotes the class of $q(x)$ in $K_1(B)$.

We can now state the problem as it was posed to us by Gert Pedersen. If $x \in M(A \otimes \mathcal{K})$ is a normal Fredholm element, is index(x) necessarily 0?

Let $\mathbf{D}$ denote the open unit disk in the complex plane. We shall show that for $A = C_0(\mathbf{D})$ and $B = C(S^2)$ there is a normal element of B that is invertible modulo A and has nonzero index. More generally, we shall show that the answer to the above problem in $M(A \otimes \mathcal{K})$ is negative whenever A is a C^*-algebra for which there exists a *-homomorphism $\varphi\colon C_0(\mathbf{D}) \to A \otimes \mathcal{K}$ such that $\varphi_* \neq 0$ in K_0. Moreover in Theorem 2 we shall show that this is the only way in which $M(A \otimes \mathcal{K})$ can contain a normal Fredholm element with nonzero index.

We will use the following notations. If A is a (possibly nonunital) C^*-algebra we will write $\widetilde{A}$ for its unitalization, $M(A)$ for its multiplier algebra, and $M_2(A)$ for the algebra of 2×2 matrices with entries in A. We will let q denote the quotient map of C^*-algebras, in which case the corresponding algebra and ideal should be clear from the context.

Let us recall some definitions we shall need for our computations. Let $0 \to A \to B \to B/A \to 0$ be a short exact sequence of C^*-algebras. Suppose that $x \in B$, $\|x\| \leq 1$, and $1 - x^*x$ and $1 - xx^*$ belong to A. Let us compute index(x). Let

$$u_x = \begin{pmatrix} x & -(1 - xx^*)^{1/2} \\ (1 - x^*x)^{1/2} & x^* \end{pmatrix} \quad \text{in } M_2(B).$$

Then

$$q(u_x) = \begin{pmatrix} q(x) & 0 \\ 0 & q(x)^{-1} \end{pmatrix}.$$

So

$$\partial[q(x)] = \left[u_x \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_x^{-1}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = [p_x] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right],$$

$$\text{where } p_x = \begin{pmatrix} xx^* & x(1 - x^*x)^{1/2} \\ x^*(1 - xx^*)^{1/2} & 1 - x^*x \end{pmatrix} \in M_2(\widetilde{A})$$

(see Blackadar [2, 8.3.1]). We remark that in fact $p_x \in \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + M_2(A)$.

The following example is of particular interest: $0 \to C_0(\mathbf{D}) \to C(\overline{\mathbf{D}}) \to C(S^1) \to 0$. Let $y \in C(\overline{\mathbf{D}})$ be given by $y(z) = z$. Then $\partial[q(y)] = [p] - \left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right]$, where $p \in M_2(C(\overline{\mathbf{D}}))$ is the projection

$$p(z) = \begin{pmatrix} |z|^2 & z(1 - |z|^2)^{1/2} \\ \overline{z}(1 - |z|^2)^{1/2} & 1 - |z|^2 \end{pmatrix} \quad \text{for } z \in \overline{\mathbf{D}}.$$

For each z, $p(z)$ is the projection onto the subspace of $\mathbf{C}^2$ spanned by $\left(\begin{smallmatrix} z \\ (1-|z|^2)^{1/2} \end{smallmatrix}\right)$. Now p is constant on $\partial\overline{\mathbf{D}}$, and thus defines a projection-valued function on $\widetilde{\mathbf{D}} = S^2$. It is easily seen that the corresponding vector bundle over

S^2 is obtained from trivial bundles on the two hemispheres, glued together along the equator by the map z. Since the one-dimensional vector bundles over S^2 are classified by the winding number of the gluing map, we see that the *Bott element* $b = [p] - [\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)]$ is a generator for $K_0(C_0(\mathbf{D}))$. Thus y is a normal Fredholm element of $C(\overline{\mathbf{D}})$ with nonzero index b.

To extend this example to other C^*-algebras, we require a lemma.

Lemma. *Let A and B be C^*-algebras, let $h\colon A \to B$ be a *-homomorphism, and suppose that $\widetilde{h}\colon M(A) \to M(B)$ is a *-homomorphism extending h. Let $y \in M(A)$ be essentially unitary (modulo A), and put $x = \widetilde{h}(y) + 1 - \widetilde{h}(1)$. Then x is essentially unitary (modulo B), and $\partial[q(x)] = h_* \circ \partial[q(y)]$.*

Remark. If $\widetilde{h}$ is unital, this says that $\partial[q \circ \widetilde{h}(y)] = h_* \circ \partial[q(y)]$.

Proof. It is easily verified that x is an essentially unitary element of $M(B)$. Perturbing y by an element of A, we may assume that $||y|| = 1$, and thus that $||x|| = 1$. Then

$$\begin{aligned}\partial[q(x)] &= [p_x] - \left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]\\ &= \left[\begin{pmatrix}1&0\\0&0\end{pmatrix} + h\left\{p_y - \begin{pmatrix}1&0\\0&0\end{pmatrix}\right\}\right] - \left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]\\ &= h_*\left\{[p_y] - \left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]\right\}\\ &= h_* \circ \partial[q(y)]. \quad \square\end{aligned}$$

As a consequence we have the following result.

Proposition. *Let A be a C^*-algebra, let $\varphi\colon C_0(\mathbf{D}) \to A$ be a *-homomorphism with $\varphi_* \neq 0$ in K_0, and suppose that φ preserves approximate units. Then there is a normal essentially unitary element x in $M(A)$ with (nonzero) index $\varphi_*(b)$.*

Proof. Arguing as in 3.12.12 of [7], φ induces a unital *-homomorphism $\widetilde{\varphi}\colon M(C_0(\mathbf{D})) \to M(A)$. Let $x = \widetilde{\varphi}(y)$. It follows from the lemma that x is normal and essentially unitary, and that $\operatorname{index}(x) = \varphi_*(\operatorname{index}(y)) = \varphi_*(b)$. $\square$

If $\varphi\colon C_0(\mathbf{D}) \to A$ does not preserve approximate units but $\varphi_* \neq 0$ in K_0, we would like to produce a normal element in $M(A \otimes \mathscr{K})$ which is essentially unitary (modulo $A \otimes \mathscr{K}$) and has nonzero index. For example, we wish to find an essentially unitary normal element of $M(C(S^2) \otimes \mathscr{K})$ with nonzero index. To this end we have the following result, which relies on a lemma of N. Higson.

Theorem 1. *Let A be a C^*-algebra and let $\varphi\colon C_0(\mathbf{D}) \to A$ be a *-homomorphism such that $\varphi_* \neq 0$ in K_0. Then there is a normal essentially unitary element in $M(A \otimes \mathscr{K})$ with index $\varphi_*(b)$, where we identify the K-theory of A and $A \otimes \mathscr{K}$, and b is the Bott element.*

Proof. Note first that $C_0(\mathbf{D})$ is an essential ideal in $C_0(\mathbf{D})^\sim$, and that $C_0(\mathbf{D})$ is σ-unital. Thus the proof of Lemma 1.4 of [5] provides a path v_t of isometries in $M(C_0(\mathbf{D}) \otimes \mathscr{K})$ with the properties:

(i) $v_0 \in 1 \otimes M(\mathscr{K})$, and
(ii) $(C_0(\mathbf{D})^\sim \otimes \mathscr{K})v_1 \subseteq C_0(\mathbf{D}) \otimes \mathscr{K}$.

From (ii) it follows that $v_1 M(C_0(\mathbf{D}) \otimes \mathscr{K})v_1^* \subseteq M(C_0(\mathbf{D})^\sim \otimes \mathscr{K})$; hence $\mathrm{Ad}(v_1)$ gives a (nonunital) *-homomorphism: $M(C_0(\mathbf{D}) \otimes \mathscr{K}) \to M(C_0(\mathbf{D})^\sim \otimes \mathscr{K})$ which carries $C_0(\mathbf{D}) \otimes \mathscr{K}$ into itself.

Let $\varphi_1 : C_0(\mathbf{D})^\sim \to \widetilde{A}$ be the unitalization of the given map φ. Let φ_1 also denote the obvious induced map $\varphi_1 : C_0(\mathbf{D})^\sim \otimes \mathscr{K} \to \widetilde{A} \otimes \mathscr{K}$. Note that φ_1 is σ-unital, and hence induces a unital map $\widetilde{\varphi_1} : M(C_0(\mathbf{D})^\sim \otimes \mathscr{K}) \to M(\widetilde{A} \otimes \mathscr{K})$. Note that $\widetilde{\varphi_1}(C_0(\mathbf{D}) \otimes \mathscr{K}) \subseteq A \otimes \mathscr{K}$. Hence we obtain the composite map $\widetilde{h} = \widetilde{\varphi_1} \circ \mathrm{Ad}(v_1) : M(C_0(\mathbf{D}) \otimes \mathscr{K}) \to M(\widetilde{A} \otimes \mathscr{K}) \subseteq M(A \otimes \mathscr{K})$, and observe that $\widetilde{h}(C_0(\mathbf{D}) \otimes \mathscr{K}) \subseteq A \otimes \mathscr{K}$. Let $h = \widetilde{h}|C_0(\mathbf{D}) \otimes \mathscr{K} : C_0(\mathbf{D}) \otimes \mathscr{K} \to A \otimes \mathscr{K}$, and observe further that $h = (\varphi \otimes 1) \circ \mathrm{Ad}(v_1)$. It follows from (i) that $\mathrm{Ad}(v_1)_*$ is the identity map in K-theory. Thus $h_* = \varphi_*$.

Now let $\widetilde{y} = y \otimes e + 1 \otimes (1 - e)$ in $M(C_0(\mathbf{D}) \otimes \mathscr{K})$, where $e \in \mathscr{K}$ is a rank one projection, and $y \in C(\overline{\mathbf{D}})$ is the element discussed earlier whose index is b. Then $\widetilde{y}$ is essentially unitary, and it is easy to see that $\partial[q(\widetilde{y})] = b$. Let $x = \widetilde{h}(\widetilde{y}) + 1 - \widetilde{h}(1)$. By the lemma, x is essentially unitary, and

$$\mathrm{index}(x) = \partial[q(x)] = h_* \circ \partial[q(\widetilde{y})] = \varphi_*(b). \quad \square$$

We now give some examples.

Example 1. Let $A = C_0(\mathbf{D})^\sim = C(S^2)$, and let φ be the inclusion map. Then the theorem implies that there is a normal essentially unitary element in $M(C(S^2) \otimes \mathscr{K})$ with nonzero index.

We remark that this example does not answer the question we raised in the second paragraph of this paper. This is because the multiplier algebra $M(C(S^2) \otimes \mathscr{K})$ is equal to the algebra of functions from S^2 into $B(H)$ which are continuous relative to the strong-* (equivalently σ-strong-*) topology [2, 12.1.1], rather than the norm topology. However, a theorem of Pimsner, Popa, and Voiculescu, [8, Theorem 3.3], allows us to conclude that there is a norm-continuous example.

Let $T_0 : S^2 \to B(H)$ be the strong-* continuous map we have shown to exist. So $T_0(\zeta)$ is normal and essentially unitary for each $\zeta \in S^2$, and the index of T_0 is nonzero. Let $U \in B(H)$ be any unitary with full spectrum, and T_1 be the function on S^2 with constant value U. Then $T_2 = T_0 \oplus T_1$ will serve as well as T_0 as an example of a normal essentially unitary element of $M(C(S^2) \otimes \mathscr{K})$ with nonzero index. Now T_2 satisfies the homogeneity requirement of [8, Theorem 3.3]. Thus there is a norm-continuous map $T : S^2 \to B(H)$ that is unitarily equivalent to T_2 at each point of S^2 and that defines an element in

Ext $(C(S^1), C(S^2))$ equivalent to that defined by T_2. Hence T is normal and essentially unitary, and has the same (nonzero) index as T_0.

Example 2. By [6] there is a *-homomorphism φ from $C_0(\mathbf{D})$ into an AF algebra A which induces an injection in K-theory. Then Theorem 1 implies that there is a normal essentially unitary element in $M(A \otimes \mathscr{K})$ with nonzero index.

Example 3. Recent work of Elliott and Loring allows us to refine Example 2. We may assume (by [3]) that A is a simple unital (separable) AF algebra.

Finally, we give a converse to the above construction. Namely, this construction is the only way in which a normal Fredholm element can have nonzero index

Theorem 2. *Let $0 \to A \to B \to B/A \to 0$ be an exact sequence of C^*-algebras. Suppose that x is a normal element of B which is invertible modulo A and such that $\partial[q(x)] \neq 0$ in $K_0(A)$. Then there is a normal element u in B such that $q(u)$ is unitary and*

(i) $\mathrm{sp}(u) = \overline{\mathbf{D}}$;
(ii) $\mathrm{sp}(q(u)) = S^1$;
(iii) *there is a path of normal element in B, each invertible modulo A, connecting u to x.*

Moreover, the *-homomorphism $\varphi\colon C_0(\mathbf{D}) \to A$ given by $\varphi(f) = f(u)$ satisfies $\varphi_*(b) = \partial[q(x)]$, where $b \in K_0(C_0(\mathbf{D}))$ is the Bott element.

Proof. Choose numbers $0 < \delta < \varepsilon < 1$ such that $|z| > \varepsilon$ for $z \in \mathrm{sp}(q(x))$. Let r be a positive continuous function on $[\delta, \varepsilon]$ which increases from 1 to ε^{-1}. Let $g(z)$ be defined for $z \in \mathbf{C}$ by

$$g(z) = \begin{cases} z, & |z| < \delta, \\ r(|z|)z, & \delta \le |z| \le \varepsilon, \\ z|z|^{-1}, & \varepsilon < |z|. \end{cases}$$

Note that the restriction of g to any compact subset of $\mathbf{C}$ is homotopic to the identity function $z \mapsto z$, via a path of continuous functions which are all equal to g on the set $\{z \in \mathbf{C}\colon |z| < \delta\}$, and are all nonzero for $|z| \ge \delta$. Let $u = g(x)$. By the definition of g it follows that $\mathrm{sp}(u) \subseteq \overline{\mathbf{D}}$. If $\mathrm{sp}(u) \neq \overline{\mathbf{D}}$ then there is a retraction of $\mathrm{sp}(u)$ to S^1. It follows that u is homotopic (via essentially invertible elements) to a unitary. Since $\partial[q(u)] = \partial[q(x)] \neq 0$, this is impossible. Thus (i) holds. It is clear from the definition of g that $q(u)$ is unitary, and in fact is the unitary part of the polar decomposition of $q(x)$. Thus (iii) holds. If $\mathrm{sp}(q(u)) \neq S^1$ then $q(u)$ is homotopic to 1 within the unitary group of B/A, again violating $\partial[q(u)] \neq 0$. Thus (ii) holds. Finally, define a map $\varphi\colon C(\overline{\mathbf{D}}) \to B$ by $\varphi(f) = f(u)$. If $f \in C_0(\mathbf{D})$ then $q(\varphi(f)) = q(f(u)) = f(q(u)) = 0$, so $\varphi(f) \in A$. Thus $\varphi\colon C_0(\mathbf{D}) \to A$, and $\partial[q(x)] = \partial[q(u)] = \varphi_*(\partial[q(y)]) = \varphi_*(b)$. □

We may now summarize the results of Theorems 1 and 2 in the following way.

Corollary. *Let A be a C^*-algebra. The following are equivalent*:

(i) *There is a* **-homomorphism* $\varphi: C_0(\mathbf{D}) \to A \otimes \mathcal{K}$ *with* $\varphi_* \neq 0$ *in* K_0.
(ii) *There is a normal Fredholm element in* $M(A \otimes \mathcal{K})$ *with nonzero index.*

Remark. In [4] a filtration of K-theory is defined for which the second filtration F_2 is the set of all images of b under maps φ_* as in (i). Thus the corollary implies that the set of indices of normal Fredholm elements of $M(A \otimes \mathcal{K})$ is precisely $F_2(A)$.

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