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A CHARACTERIZATION OF SPECTRAL OPERATORS

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ABSTRACT. A characterization of spectral operators due to N. Dunford is simplified. Especially, his complicated Condition (D) is replaced by a very simple one.

Dunford [2; 3; p. 2147] showed that an operator T defined on a weakly complete Banach space Y is spectral if and only if it satisfies the following conditions (i)–(iv).

(i) Dunford's analyticity condition (A). T has the single-valued extension property; that is, for every $x \in Y$ there exists a smallest closed set $\sigma_T(x)$, called the local spectrum of T at x, and a unique *analytic* function $f: \mathbf{C}\backslash\sigma_T(x) \to Y$, called the local resolvent of T at x, such that $(z - T)f(z) \equiv x$.

(ii) Dunford's boundedness condition (B). There exists a *bound* K such that $\|x\| \le K\|x + y\|$ whenever $\sigma_T(x) \cap \sigma_T(y) = \varnothing$.

(iii) Dunford's closure condition (C). For every closed set $F \subset \mathbf{C}$, the set $Y_T(F) := \{x: \sigma_T(x) \subset F\}$ is *closed*.

(iv) Dunford's decomposability condition (D). Let $\mathscr{S}_1(T)$ denote the collection of all $\delta \subset \mathbf{C}$ for which $Y_T(\delta) + Y_T(\mathbf{C} \setminus \delta)$ is dense in Y. Assuming T satisfies condition (B), it follows that Y is the direct sum of $\overline{Y_T(\delta)}$ and $\overline{Y_T(\mathbf{C} \setminus \delta)}$ and the corresponding projections $E(\delta)$ and $E(\mathbf{C} \setminus \delta)$ are bounded by K. Let $\mathscr{S}_2(T)$ denote the collection of all $\delta \in \mathscr{S}_1(T)$ such that for every $\varepsilon > 0$ there exist $x_1, x_2 \in Y$ with $\|x_1 + x_2 - x\| < \varepsilon$, $\sigma_T(x_1) \subset \delta \cap \sigma_T(x)$, and $\sigma_T(x_2) \subset \sigma_T(x) \setminus \delta$. We assume that for every $\lambda \in \mathbf{C}$ and $r > 0$ there exist $\delta, \mu_n, \gamma_n \in \mathscr{S}_2(T)$ $(n = 1, 2, \dots)$ such that $\lambda \in \delta^0$, $\operatorname{diam}(\delta) \le r$, $\mu_n \subset \delta$, $\gamma_n \subset \mathbf{C} \setminus \delta$, μ_n closed, γ_n closed $(n = 1, 2, \dots)$, and $x = \lim[E(\mu_n)x + E(\gamma_n)x]$.

The purpose of this paper is to simplify the above criterion. For an arbitrary operator T, and a subset δ of $\mathbf{C}$, we define $S_T(\delta)$ to be the set of all $x \in Y$ such that $(z - T)f(z) \equiv x$ for some analytic Y-valued function f defined on an open set containing $\mathbf{C} \setminus \delta$.

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Theorem. *A (bounded linear) operator T on a weakly complete Banach space Y is spectral if and only if T satisfies the following conditions.*

(a) *There exists $K > 0$ such that $\|x\| \leq K\|x+y\|$ whenever $x \in S_T(\delta)$, $y \in S_T(\mathbf{C} \setminus \delta)$, and $\delta = \overline{\delta} \subset \mathbf{C}$.*

(b) *$S_T(\Delta)$ is closed and $S_T(\Delta) + S_T(\mathbf{C} \setminus \Delta)$ is dense in Y whenever Δ is a closed (vertical or horizontal) half plane.*

Proof. We first prove the sufficiency of the conditions. Assume the equation $(z-T)f(z) \equiv 0$ has a nonzero analytic solution f. Let $N(\lambda; r)$ be an arbitrary open disc lying entirely in the domain of f. Since $(z-T)(z-\lambda)^{-1}f(\lambda) \equiv f(\lambda)$ for $z \neq \lambda$ it follows that $f(\lambda) \in S_T(\{\lambda\})$. Define $g(z) = (z-\lambda)^{-1}[f(z) - f(\lambda)]$ for $z \in N(\lambda; r) \setminus \{\lambda\}$, and $g(\lambda) = f'(\lambda)$ [1, p. 2]. Then $(z-T)g(z) \equiv -f(\lambda)$ for $z \in N(\lambda; r)$ and thus $-f(\lambda) \in S_T(\mathbf{C} \setminus N(\lambda; r))$. In view of condition (B), $\|f(\lambda)\| \leq K\|f(\lambda) - f(\lambda)\| = 0$. Since λ is arbitrary, $f \equiv 0$. Thus T satisfies condition (A) and $S_T(\bigcap F_\alpha) = \bigcap S_T(F_\alpha)$ for any family $\{F_\alpha\}$ of subsets of $\mathbf{C}$. Also in view of (a), T satisfies condition (B).

Now, let $\delta \subset \mathbf{C}$ be such that $Y = \overline{S}_T(\delta) \oplus \overline{S}_T(\mathbf{C} \setminus \delta)$. Define $E(\delta)$ to be the projection onto $\overline{S}_T(\delta)$ that is parallel to $\overline{S}_T(\mathbf{C} \setminus \delta)$. Because $(z-T)E(\delta)f(z) \equiv E(\delta)x$ whenever f is the local resolvent of T at x, it follows that $\sigma_T(E(\delta)x) \subset \sigma_T(x) \cap \overline{\delta}$ provided that $S_T(\overline{\delta})$ is closed. Moreover, $\|E(\delta)\| \leq K$, $E(\delta_1)E(\delta_2) = E(\delta_1)$ if $\delta_1 \subset \delta_2$, and $E(\delta_1)E(\delta_2) = 0$ if $\delta_1 \cap \delta_2 = \varnothing$. In particular, $E(\Delta)$ and $E(\mathbf{C} \setminus \Delta)$ are defined for any closed (vertical or horizontal) half plane Δ.

We claim $E(R)$ can be defined for any semiclosed rectangle R of the form $(a, b] \times (c, d]$. Let $x \in Y$. Let Δ be as in (b) and assume $\{\Delta_n\}$ is a strictly decreasing sequence of closed half planes converging to Δ. Because $u_n = E(\mathbf{C} \setminus \Delta)\ E(\Delta_n)x$ is bounded, every subsequence of $\{u_n\}$ has a weakly convergent subsequence. Let u be the limit of any weakly convergent subsequence of $\{u_n\}$. Since $u_n \in S_T(\Delta_N) \cap \overline{S}_T(\mathbf{C} \setminus \Delta)$ for all $n \geq N$, it follows that $u \in S_T(\Delta_N) \cap \overline{S}_T(\mathbf{C} \setminus \Delta)$ and hence $u \in S_T(\Delta) \cap \overline{S}_T(\mathbf{C} \setminus \Delta) = \{0\}$. Thus $v_n = E(\mathbf{C}\setminus\Delta)E(\mathbf{C}\setminus\Delta_n)x$ converges weakly to $E(\mathbf{C}\setminus\Delta)x$. Therefore, a sequence w_n of finite convex combinations of v_n converges strongly to $E(\mathbf{C}\setminus\Delta)x$. Since $\sigma_T(v_n) \subset \sigma_T(x) \setminus \Delta$, $\sigma_T(w_n) \subset \sigma_T(x) \setminus \Delta$ for all n. Summing up, we have shown that for every $x \in Y$, Δ as in (b), and $\varepsilon > 0$, there exists $w \in Y$ such that $\|E(\mathbf{C} \setminus \Delta)x - w\| < \varepsilon$ and $\sigma_T(w) \subset \sigma_T(x) \setminus \Delta$. Thus by consecutive applications of this result to the half planes $\{z\colon \operatorname{Re} z \leq a\}$, $\{z\colon \operatorname{Re} z \leq b\}$, $\{z\colon \operatorname{Im} z \leq c\}$, and $\{z\colon \operatorname{Im} z \leq d\}$, we obtain vectors x_1 and $x_2 \in Y$ such that $\sigma_T(x_1) \subset \sigma_T(x) \setminus R$, $\sigma_T(x_2) \subset S_T(R)$, and $\|x - x_1 - x_2\| < \varepsilon$. Thus $S_T(R) + S_T(\mathbf{C} \setminus R)$ is dense in Y and hence $Y = \overline{S}_T(R) \oplus \overline{S}_T(\mathbf{C} \setminus R)$.

Let $R_0 = (-a, a] \times (-a, a]$ be a semiclosed square containing $\sigma(T)$. For each $n \in \mathbf{N}$, let $\{D_{nk}\colon k = 1, 2, \ldots, n\}$ be the family of disjoint semiclosed squares of diameter $a\sqrt{2}/2^n$ covering R_0. It is clear that E is a bounded additive set function on the Boolean algebra generated by $\{D_{nk}\colon k = 1, 2, \ldots, 4^n; n = 1, 2, \ldots\}$. Define $S_n = \sum_{k=1}^{4^n} z_{nk}E(D_{nk})$, where z_{nk} is the centre of D_{nk}. For $m > n$, we have

$$\|S_m - S_n\| = \left\| \sum_{k=1}^{4^n} c_{mk} E(D_{mk}) \right\| \leq 4a\sqrt{2}/2^n ,$$

where c_{mk} is the difference between the centre of D_{mk} and the centre of D_{nj} such that $D_{mk} \subset D_{nj}$ [3; p. 2181]. Hence $\{S_n\}$ converges to a scalar-type spectral operator A [3; p. 2192]. Obviously, $AT = TA$. We claim $T - A$ is quasinilpotent. Let T_{nk} and A_{nk} be the restrictions of T and A to $E(D_{nk})Y = \overline{S}_T(D_{nk})$ for all possible pairs (n, k). Since $S_T(\overline{D}_{nk})$ is the intersection of four spectral manifolds corresponding to closed half planes, it follows that $S_T(\overline{D}_{nk})$ is closed and hence $\sigma(T|\overline{S}_T(D_{nk})) \subset \sigma(T|S_T(\overline{D}_{nk})) \subset \overline{D}_{nk}$ [1, p. 23]. Also, let $S_{mnk} = S_m|\overline{S}_T(D_{nk})$ and observe that $\|(z - S_{mnk})^{-1}\| \leq K/\operatorname{dist}(z, D_{nk})$. Letting $m \to \infty$ it yields $\|(z - A_{nk})^{-1}\| \leq K \operatorname{dist}/(z, D_{nk})$ for all possible (n, k). Thus $\sigma(A_{nk}) \subset \overline{D}_{nk}$ and hence $\sigma(T_{nk} - A_{nk}) \subset \{\alpha - \beta : \alpha \in \overline{D}_{nk}, \beta \in \overline{D}_{nk}\}$ [8, Chapter 0]. Therefore, $\sigma(T_{nk} - A_{nk})$ lies in a disc of radius $a\sqrt{2}/2^{n-1}$ for all possible (n, k). It follows that

$$\sigma(T - A) \subset \bigcup_{k=1}^{4^n} \sigma(T_{nk} - A_{nk}) \subset \{\lambda : |\lambda| < a\sqrt{2}/2^{n-1}\}$$

for all n, and thus $\sigma(T - A) = \{0\}$. Therefore, $T - A$ is quasinilpotent and consequently, T is spectral.

The necessity of the conditions is an immediate consequence of Dunford's characterization. □

Remarks. (1) We have shown that if T satisfies condition (a) of the theorem, then it necessarily satisfies condition (A).

(2) A sort of strong regularity of E depicted by $x = \lim[E(\mu_n) + E(\gamma_n)]x$ required in Dunford's condition (D) is not needed in our criterion. We at most require $S_T(\Delta) + S_T(\mathbf{C} \setminus \Delta)$ to be dense in Y. In other words, we need only the closed (vertical or horizontal) half planes to belong to $\mathscr{S}_1(T)$.

(3) Dunford's requirement that $S_T(F)$ be closed for every set F is reduced in our criterion to the requirement that $S_T(\Delta)$ be closed for all closed half planes. The following example shows that T may fail to be spectral if, in Condition (b) of the theorem, Δ is assumed only to be a closed right or upper half plane.

(4) Some of the ideas are taken from the proof of Theorem 1 of [7].

Example. Let V be a nonunitary contraction operator on a Hilbert space K with $\sigma(B) = \{1\}$ [5, Problem 150]. Let φ be a conformal mapping from the unit disc onto $\Delta_1 = \{re^{i\theta} : 0 \leq r \leq 1, 0 \leq \theta \leq \pi/4\}$ such that $\varphi(1) = 0$. Let $A = \varphi(V)$. Then $\sigma(A) = \{0\}$ and Δ_1 is a spectral set for A [4, §1.1; 9, p. 143; 6, proof of Proposition 1]. Then the set $\Delta_n = \{re^{i\theta} : 0 \leq r \leq 1, 0 \leq \theta \leq \pi/4n\}$ is a spectral set for $A^{1/n} = \varphi_n(A)$, where $\varphi_n(re^{i\theta}) = r^{1/n}e^{i\theta/n}$

and $re^{i\theta} \in \Delta_n$ $(n \in \mathbf{N})$. Let $T = A \oplus A^{1/2} \oplus \cdots$. As it is shown in the proof of Proposition 1 of [6], $\sigma(T) = [0, 1]$, $\overline{S}_T(\{0\}) = H$, and T satisfies the single-valued extension property where $H = K \oplus K \oplus \cdots$. Thus $S_T(\{0\})$ is not closed [1; p. 23]. Hence, in the light of Dunford's criterion, T is not spectral. Let $0 \neq x \in H$ be arbitrary. We claim $0 \in \sigma_T(x)$. Let f be the local resolvent of T at x and assume, if possible, that $0 \notin \sigma_T(x)$. Let x_n be the projection of x in the nth copy of K. Then $\sigma_T(x_n) = \sigma_{A^{1/n}}(x_n) \subset \sigma_T(x)$. Since $A^{1/n}$ is quasinilpotent, $x_n = 0$ $(n \in \mathbf{N})$. Thus $x = 0$, a contradiction. Thus condition (a) holds trivially. Moreover, if δ is any Borel set, then $S_T(\delta) + S_T(\mathbf{C} \setminus \delta)$ is dense in H because it contains $S_T(\{0\})$. Now, let Δ be any closed horizontal half plane. Then, either $[0, 1] \subset \Delta$, in which case $S_T(\Delta) \supset S_T(\sigma(T)) = H$, or $0 \notin \Delta$, from which follows $S_T(\Delta) = \{0\}$. Finally, assume Δ is a closed right half plane. Then $S_T(\Delta) = \{0\}$ if $0 \notin \Delta$, and $S_T(\Delta) = H$ if $0 \in \Delta$.

Added in proof. We are grateful to the referee who brought to our attention a preprint of Professor B. L. Wadhwa's talk [10], where he shows that T is spectral if and only if T satisfies the above Dunford's Conditions (A), (B), (C) and the following Condition (D$'$): $S_T(\delta) + S_T(\mathbf{C} \setminus \delta)$ is dense in Y for all closed sets δ. Thus the main result of our paper extends Wadhwa's result by omitting Condition (A) and restricting Conditions (C) and (D$'$) to closed half planes.

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