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OSCILLATIONS IN NEUTRAL EQUATIONS WITH PERIODIC COEFFICIENTS

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ABSTRACT. We obtain a necessary and sufficient condition for the oscillation of all solutions of the neutral delay differential equation:

$$\frac{d}{dt}[x(t)+px(t-\tau)]+Q(t)x(t-\sigma)=0, \tag{1}$$

where $p \in \mathbf{R}$, $Q \in C[[0,\infty), \mathbf{R}^+]$, Q is ω-periodic with $\omega > 0$, $Q(t) \not\equiv 0$ for $t \geqq 0$, and there exist positive integers n_1 and n_2 such that $\tau = n_1\omega$ and $\sigma = n_2\omega$. More precisely we show that every solution of (1) oscillates if and only if every solution of an associated neutral equation with constant coefficients oscillates.

1. INTRODUCTION AND PRELIMINARIES

Consider the neutral delay differential equation

$$\frac{d}{dt}[x(t)+px(t-\tau)]+Q(t)x(t-\sigma)=0, \tag{1}$$

where

(2) $p \in \mathbf{R}$, $Q \in C[[0,\infty), \mathbf{R}^+]$, Q is ω-periodic with $\omega > 0$, $Q(t) \not\equiv 0$ for $t \geqq 0$, and there exist positive integers n_1 and n_2 such that $\tau = n_1\omega$ and $\sigma = n_2\omega$.

Our aim in this paper is to obtain a necessary and sufficient condition for the oscillation of all solutions of (1). More precisely we will establish the following result.

Theorem 1. *Assume that* (2) *is satisfied and set*

$$\tau_1 = \int_0^\tau Q(s)\,ds \quad \text{and} \quad \sigma_1 = \int_0^\sigma Q(s)\,ds. \tag{3}$$

Then the following statements are equivalent.

(a) *Every solution of* (1) *oscillates.*

(b) *Every solution of the neutral equation with constant coefficients*

$$\frac{d}{dt}[y(t)+py(t-\tau_1)]+y(t-\sigma_1)=0 \tag{4}$$

oscillates.

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This theorem shows that the oscillatory behavior of (1) when Q is periodic and (2) holds is characterized by the oscillatory behavior of the linear autonomous neutral equation (4). On the other hand it is also known (see [12]) that every solution of (4) oscillates if and only if its characteristic equation

$$\lambda + p\lambda e^{-\lambda\tau_1} + e^{-\lambda\sigma_1} = 0 \tag{5}$$

has no real roots.

The way that we were led to Theorem 1 was by looking for a solution of (1) in the form

$$x(t) = e^{\lambda \int_0^t Q(s)\,ds}. \tag{6}$$

By substituting (6) into (1) we find

$$e^{\lambda \int_0^t Q(s)\,ds}\left[\lambda Q(t) + p\lambda Q(t-\tau)e^{-\lambda \int_{t-\tau}^t Q(s)\,ds} + Q(t)e^{-\lambda \int_{t-\sigma}^t Q(s)\,ds}\right] = 0. \tag{7}$$

Therefore if Q is not identically zero and if Q is ω-periodic with τ and σ integral multiples of ω, then (7) reduces to (5) where τ_1 and σ_1 are defined by (3). The fact that (5) characterizes the oscillatory behavior of (1) is a remarkable result that is not obvious.

The oscillatory behavior of neutral differential equations has been the subject of many recent investigations. See, for example [1]–[5], [8], and [12] and the references cited therein. The technique that we employ in the proof of Theorem 1 was initiated in [9] and [10] and had been used successfully in linear neutral autonomous equations. See, for example, [4], [5], and [13]. It is, however, surprising that the same technique may be modified to also apply to equations with periodic coefficients.

Let $\gamma = \max[\tau, \sigma\}$. By a *solution* of (1) we mean a function x where $x \in C[[t_0 - \gamma, \infty), \mathbf{R}]$, for some $t_0 \geqq 0$, such that $x(t) + px(t-\tau)$ is continuously differentiable on $[t_0, \infty)$ and (1) is satisfied for $t \geqq t_0$.

Assume that $p \in \mathbf{R}$, $\tau, \sigma \in (0, \infty)$, and $Q \in C[[0, \infty), \mathbf{R}]$. Let $t_0 \geqq 0$ be a given initial point and let $\phi \in C[[t_0 - \gamma, t_0], \mathbf{R}]$ be a given initial function. Then by the method of steps, one can see that (1) has a unique solution $x \in C[[t_0 - \gamma, \infty), \mathbf{R}]$ such that

$$x(t) = \phi(t) \quad \text{for } t_0 - \gamma \leqq t \leqq t_0.$$

As is customary, a solution of (1) is said to *oscillate* if it has arbitrarily large zeros. Otherwise the solution is called *nonoscillatory*.

In the sequel, unless otherwise specified, when we write a functional inequality we will assume that it holds for all sufficiently large values of t.

2. The proof of Theorem 1

When $p = 0$, the proof of Theorem 1 is a consequence of known results. Indeed, in this case (1) reduces to

$$\dot{x}(t) + Q(t)x(t-\sigma) = 0, \tag{1'}$$

where

$$Q \in C[[0, \infty], \mathbf{R}^+] \quad \text{and} \quad \sigma > 0.$$

Now it is well known, see [6] and [7], that every solution of $(1')$ oscillates provided that

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} Q(s)\, ds > \frac{1}{e}.$$

It is also known (see [11]) that $(1')$ has a nonoscillatory solution provided that

$$\sup_{t \geqq \sigma} \int_{t-\sigma}^{t} Q(s)\, ds \leqq \frac{1}{e}.$$

Hence if Q is σ-periodic and if we set

$$\sigma_1 = \int_0^\sigma Q(s)\, ds,$$

then every solution of $(1')$ oscillates if and only if

$$\sigma_1 > \tfrac{1}{e};$$

that is, if and only if every solution of

$$\dot{y}(t) + y(t - \sigma_1) = 0$$

oscillates (see [9]).

In view of the above discussion, in the sequel, we will assume that

$$p \neq 0.$$

For the proof of (a) $\Rightarrow$ (b) assume, for the sake of contradiction, that (4) has a nonoscillatory solution. Then (5) must have a real root λ_0 (see [12]). Now by direct substitution into (1), one can see that

$$x(t) = e^{\lambda_0 \int_0^t Q(s)\, ds} \quad \text{for } t \geqq \max\{\tau, \sigma\}$$

is a nonoscillatory solution of (1). This contradicts the hypothesis that every solution of (1) oscillates and completes the proof that (a) $\Rightarrow$ (b).

The proof that (b) $\Rightarrow$ (a) is quite involved and will be accomplished by establishing a series of lemmas.

In the sequel we will assume that (b) holds and that, for the sake of contradiction, (1) has an eventually positive solution that we will denote by $x(t)$. The proof will be completed when we reach a contradiction.

Set

$$F(\lambda) = \lambda + p\lambda e^{-\lambda\tau_1} + e^{-\lambda\sigma_1}.$$

The hypothesis that (b) holds is equivalent to the fact that (5) has no real roots (see [12]). As $F(0) = 1$, it follows that

$$F(\lambda) > 0 \quad \text{for } \lambda \in \mathbf{R}.$$

Also $F(\infty) = \infty$ and clearly $F(-\infty)$ must be ∞, for, otherwise, $F(-\infty) = -\infty$ and so (5) would have a real root.

The following lemma is now an elementary consequence of the above observations.

Lemma 1. *The following statements are true.*

(a) *There exists a positive number* m_0 *such that*

$$\lambda + p\lambda e^{-\lambda\tau_1} + e^{-\lambda\sigma_1} \geqq m_0 \quad \text{for } \lambda \in \mathbf{R}$$

or equivalently

$$\lambda + p\lambda e^{\lambda\tau_1} - e^{\lambda\sigma_1} \leqq -m_0 \quad \text{for } \lambda \in \mathbf{R}.$$

(b) *If* $p > 0$ *then* $\sigma_1 > \tau_1$ *and* $\sigma > \tau$.

The following lemma can easily be proved by direct substitution into (1) and its proof will be omitted.

Lemma 2. *If* $v(t)$ *is a solution of* (1) *for* $t \geqq t_0 \geqq 0$ *then*

$$w_1(t) = v(t) + pv(t-\tau) \quad \text{for } t \geqq t_0 + \tau$$

and

$$w_2(t) = \int_{t-\sigma}^{t-\tau} Q(s)v(s) \quad \text{for } t \geqq t_0 + \max\{\tau, \sigma\}$$

are also solutions of (1).

Before we state the next lemma we introduce some notation.

Let W^0 be the set of all continuously differentiable solutions $w(t)$ of (1) with the properties that

(8) $$w(t) > 0 \text{ and } \dot{w}(t) \leqq 0 \quad \text{for all large } t \text{ and } \lim_{t\to\infty} w(t) = 0.$$

Also let W^∞ be that set of all continuously differentiable solutions $w(t)$ of (1) such that

(9) $$w(t) > 0 \text{ and } \dot{w}(t) \geqq 0 \quad \text{for all large } t \text{ and } \lim_{t\to\infty} w(t) = \infty.$$

For each $w \in W^\infty$ we define the set

$$\wedge^0(w) = \{\lambda \in \mathbf{R}^+ : \dot{w}(t) + \lambda Q(t)w(t) \leqq 0 \text{ for all large } t\},$$

and for each $w \in W^0$ we define the set

$$\wedge^\infty(w) = \{\lambda \in \mathbf{R}^+ : -\dot{w}(t) + \lambda Q(t)w(t) \leqq 0 \text{ for all large } t\}.$$

Clearly $0 \in \wedge^0(w)$ for every $w \in W^0$ and $0 \in \wedge^\infty(w)$ for every $w \in W^\infty$. It is also easy to see that for any $w \in W^0 \cup W^\infty$, $\wedge^0(w)$ and $\wedge^\infty(w)$ are subintervals of $\mathbf{R}^+$.

The next three lemmas describe some interesting facts about the sets W^0 and W^∞ and the sets $\wedge^0 w$ and $\wedge^\infty(w)$ with $w \in W^0 \cup W^\infty$.

Lemma 3. $W^0 \cup W^\infty \neq \varnothing$. *That is, there exists a solution* w *of* (1) *that satisfies either* (8) *or* (9).

Proof. Set

$$y(t) = x(t) + px(t-\tau),$$

where $x(t)$ is an eventually positive solution of (1). Then

$$\dot{y}(t) = -Q(t)x(t-\sigma).$$

As $Q(t) \geqq 0$ and $Q(t) \not\equiv 0$, it follows that $\dot{w}(t) \leqq 0$ and so either eventually $y(t) > 0$ or eventually $y(t) < 0$.

First assume that eventually $y(T) > 0$. Then we claim that $y \in W^0$. Otherwise

(10) $$\lim_{t\to\infty} y(t) = l > 0.$$

Set

$$v(t) = y(t) + py(t-\tau).$$

Then

$$\dot{v}(t) = -Q(t)y(t-\sigma)$$

and so by (10) we obtain

(11) $$\dot{v}(t) \leqq -\tfrac{l}{2}Q(t).$$

Clearly $\int_0^\infty Q(s)\,ds = \infty$ and so by integrating (11) from t_0 to ∞, with t_0 sufficiently large, we are led to a contradiction.

Next assume that eventually $y(t) < 0$. Set

$$z(t) = -y(t)$$

and observe that

$$\dot{z}(t) = Q(t)x(t-\sigma).$$

Hence $z(t)$ is eventually positive and increasing and so either

(12) $$\lim_{t\to\infty} z(t) = \infty$$

or

(13) $$\lim_{t\to\infty} z(t) = L \in (0, \infty).$$

If (12) holds then $z \in W^\infty$. On the other hand, if (13) holds, set

$$u(t) = -[z(t) + pz(t-\tau)]$$

and observe that eventually

$$\dot{u}(t) = Q(t)z(t-\sigma) \geqq (L/2)Q(t).$$

From this it is easily seen that $u \in W^\infty$. The proof of Lemma 3 is complete.

When $W^0 \neq \varnothing$, the next lemma shows how to construct a sequence of functions (w_n) in W^0 and a positive number λ^* such that

$$\lambda^* \in \wedge^0(w_n) \quad \text{for all } n.$$

Also a similar construction is shown when $W^\infty \neq \varnothing$.

Lemma 4. *The following statements hold.*

(a) *Assume that* $W^0 \neq \varnothing$. *Then*

$$p > -1. \tag{14}$$

Let $w_0 \in W^0$ *and set* $w(t) = w_0(t) + pw_0(t-\tau)$. *Then* $w \in W^0$. *Furthermore,* $1 \in \Lambda^0(w)$ *if* $-1 < p < 0$ *and* $1/(1+p) \in \Lambda^0(w)$ *if* $p > 0$.

(b) *Assume that* $W^\infty \neq \varnothing$. *Then*

$$p < -1. \tag{15}$$

Let $w_\infty \in W^\infty$ *and set* $w(t) = -[w_\infty(t) + pw_\infty(t-\tau)]$ *if* $\sigma \leqq \tau$ *and*

$$v(t) = -[w_\infty(t) + pw_\infty(t-\tau)] + \int_{t-\sigma}^{t-\tau} Q(s)w_\infty(s)\,ds \quad \text{if } \sigma > \tau. \tag{16}$$

Then $w, v \in W^\infty$. *Furthermore,* $-1/p \in \Lambda^\infty(w)$ *if* $\sigma \leqq \tau$ *and* $\frac{1}{-p+\sigma_1-\tau_1} \in \Lambda^\infty(v)$ *if* $\sigma > \tau$.

Proof. (a) We have

$$\dot{w}(t) = -Q(t)w_0(t-\sigma) \leqq 0$$

and $\dot{w}(t)$ is not identically zero for all large t. Also

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty}[w_0(t) + pw_0(t-\tau)] = 0.$$

Thus $w(t)$ decreases to zero which implies that eventually $w(t) > 0$. Hence $w \in W^0$. Now observe that

$$0 < w(t) = w_0(t) + pw_0(t-\tau) \leqq (1+p)w_0(t-\tau)$$

which implies that (14) holds.

First, assume that $-1 < p \leqq 0$. Then

$$w(t) = w_0(t) + pw_0(t-\tau) < w_0(t)$$

and so

$$\dot{w}(t) = -Q(t)w_0(t-\sigma) \leqq -Q(t)w(t-\sigma) \leqq -Q(t)w(t)$$

which shows that $1 \in \Lambda^0(w)$.

Next, assume that $p > 0$. By Lemma 1(b) we also have $\sigma > \tau$. Therefore,

$$w(t) = w_0(t) + pw_0(t-\tau) \leqq (1+p)w_0(t-\tau) \leqq (1+p)w_0(t-\sigma)$$

and so

$$\dot{w}(t) = -Q(t)w_0(t-\sigma) \leqq -Q(t)w(t)/(1+p)$$

which shows that

$$1/(1+p) \in \Lambda^0(w).$$

(b) Set $z(t) = -[w_\infty(t) + pw_\infty(t-\tau)]$. Then

$$z(t) = Q(t)w_\infty(t-\sigma)$$

and one can easily see that $z \in W^{\infty}$. Hence

$$0 < z(t) = -[w_{\infty}(t) + pw_{\infty}(t-\tau)] \leqq -(1+p)w_{\infty}(t-\tau)$$

which shows that (15) holds.

First assume that $\sigma \leqq \tau$. Then $w = z$ and so $w \in W^{\infty}$. Furthermore,

$$-pw_{\infty}(t-\sigma) \geqq -pw_{\infty}(t-\tau) > w(t)$$

and so

$$0 = -\dot{w}(t) + Q(t)w_{\infty}(t-\sigma) \geqq -\dot{w}(t) + Q(t)w(t)/-p$$

which shows that

$$-1/p \in \Lambda^{\infty}(w).$$

Next assume that $\sigma > \tau$. Then from (16) we see that

$$\dot{v}(t) = Q(t)w_{\infty}(t-\tau) \tag{17}$$

from which it follows that $v \in W^{\infty}$. From (16) and (2) we see that

$$v(t) < -pw_{\infty}(t-\tau) + w_{\infty}(t-\tau)\int_{t-\sigma}^{t-\tau} Q(s)\,ds = w_{\infty}(t-\tau)(-p+\sigma_1-\tau_1). \tag{18}$$

It follows from (17) and (18) that

$$-\dot{v}(t) + \frac{1}{-p+\sigma_1-\tau_1}Q(t)v(t) \leqq 0$$

which shows that

$$\frac{1}{-p+\sigma_1-\tau_1} \in \Lambda^{\infty}(v).$$

The proof of Lemma 4 is complete.

Lemma 5. (a) *Assume that $W^0 \neq \varnothing$. Then there exists a $\lambda^* \in (0, \infty)$ such that λ^* is an upper bound of $\Lambda^0(w)$ for every $w \in W^0$.*

(b) *Assume that $W^{\infty} \neq \varnothing$. Then there exists a $\lambda^* \in (0, \infty)$ such that λ^* is an upper bound of $\Lambda^{\infty}(w)$ for every $w \in W^{\infty}$.*

Proof. (a) As we proved in Lemma 4(a), (see (14)), in this case $p > -1$.

Case 1. $-1 < p < 0$. Let $w \in W^0$. Then, clearly, $w(t) + pw(t-\tau)$ decreases to zero and so eventually

$$w(t) > -pw(t-\tau). \tag{19}$$

Let $\lambda \in \Lambda^0(w)$ and set

$$\phi_{\lambda}(t) = w(t)e^{\lambda\int_0^t Q(s)\,ds}. \tag{20}$$

Then eventually

$$\dot{\phi}_{\lambda}(t) \leqq 0$$

and so $\phi_{\lambda}(t)$ is eventually decreasing. Hence eventually,

$$w(t-\tau)e^{\lambda\int_0^{t-\tau} Q(s)\,ds} \geqq w(t)e^{\lambda\int_0^t Q(s)\,ds}$$

and in view of (3) and (19)

$$w(t-\tau) \geqq w(t)e^{\lambda \int_{t-\tau}^{t} Q(s)\,ds} = w(t)e^{\lambda\tau_1} > -pe^{\lambda\tau_1}w(t-\tau).$$

Hence

$$-pe^{\lambda\tau_1} < 1$$

which shows that

$$\lambda^* = \frac{1}{\tau_1}\ln\left(-\frac{1}{p}\right)$$

is an upper bound of $\wedge^0(w)$.

Case 2. $p > 0$. By Lemma 1(b) in this case $\sigma > \tau$. Set

$$\theta = \sigma - \tau \quad \text{and} \quad k = \int_{t-\theta}^{t} Q(s)\,ds \quad \text{for } t \geqq \theta.$$

Clearly $k > 0$. Now for every $t \geqq \theta$ let $t^* = t^*(t)$ be a point in $(t-\theta, t)$ such that

$$\int_{t-\theta}^{t^*} Q(s)\,ds = \int_{t^*}^{t} Q(s)\,ds = \frac{k}{2}.$$

Set

$$z(t) = w(t) + pw(t-\tau).$$

Then eventually $z(t) > 0$ and

$$\dot{z}(t) = -Q(t)w(t-\sigma).$$

By integrating both sides of this equation from t^* to t, for t sufficiently large, we obtain

$$z(t) - z(t^*) = -\int_{t^*}^{t} Q(s)w(s-\sigma)\,ds \leqq -w(t-\sigma)\frac{k}{2}.$$

Thus

$$(k/2)w(t-\sigma) \leqq z(t^*) - z(t) \leqq z(t^*) = w(t^*) + pw(t^*-\tau) \leqq (1+p)w(t^*-\tau)$$

and so

$$w(t^*-\tau) \geqq \frac{k}{2(1+p)}w(t-\sigma). \tag{21}$$

Let $\lambda \in \wedge^0(w)$. Now by using the fact that the function $\phi_\lambda(t)$ which we defined in (20) is decreasing and in view of (21) we find,

$$\begin{aligned} w(t-\sigma) \geqq w(t^*-\tau)e^{\lambda \int_{t-\sigma}^{t^*-\tau} Q(s)\,ds} &= w(t^*-\tau)e^{\lambda \int_{t-\theta}^{t^*} Q(s)\,ds} \\ &\geqq \frac{k}{2(1+p)}w(t-\sigma)e^{\lambda k/2}. \end{aligned}$$

Hence

$$\frac{k}{2(1+p)}e^{\lambda k/2} \leqq 1$$

which shows that

$$\lambda^* = \frac{2}{k}\ln\frac{2(1+p)}{k}$$

is an upper bound of $\wedge^0(w)$. The proof of part (a) of Lemma 5 is complete.

(b) As we proved in Lemma 4(b), (see (15)), in this case $p < -1$. Let $w \in W^\infty$ and set

$$z(t) = -[w(t) + pw(t-\tau)].$$

Then eventually $z(t) > 0$ and so

$$w(t) < -pw(t-\tau).$$

Let $\lambda \in \wedge^\infty(w)$ and set

$$\psi_\lambda(t) = w(t)e^{-\lambda\int_0^t Q(s)\,ds}. \tag{22}$$

Then eventually

$$\dot{\psi}_\lambda(t) \geqq 0$$

and so $\psi_\lambda(t)$ is eventually increasing. Hence eventually,

$$w(t) \geqq e^{\lambda\int_{t-\tau}^t Q(s)\,ds} w(t-\tau) \geqq e^{\lambda\tau_1} w(t)/(-p).$$

Therefore

$$e^{\lambda\tau_1}/(-p) \leqq 1$$

which shows that

$$\lambda^* = \ln(-p)/\tau_1$$

is an upper bound of $\wedge^\infty(w)$. The proof of Lemma 5 is complete.

On the basis of the preceding lemmas it suffices to examine each of the following four cases and in each case it remains to obtain a contradiction.

Case I: $W^0 \neq \varnothing$ and $-1 < p < 0$;
Case II: $W^0 \neq \varnothing$ and $p > 0$;
Case III: $W^\infty \neq \varnothing$ and $\sigma \leqq \tau$;
Case IV: $W^\infty \neq \varnothing$ and $\sigma > \tau$.

Our strategy in each case is to use Lemma 4 to construct a sequence of functions $\{w_n\}$ such that for each $n = 1, 2, \ldots$,

$$w_n \in W^0 \text{ in Cases I and II}$$

and

$$w_n \in W^\infty \text{ in Cases III and IV}.$$

In each case we will also find a positive number μ and show that for every $n = 1, 2, \ldots$,

$$\text{if } \lambda \in \wedge^0(w_n) \quad \text{then } \lambda + \mu \in \wedge^0(w_{n+1}) \tag{23}$$

and

$$\text{if } \lambda \in \wedge^\infty(w_n) \quad \text{then } \lambda + \mu \in \wedge^\infty(w_{m+1}). \tag{24}$$

In view of Lemma 5, (23) and (24) will eventually lead to the desired contradiction.

Case I. $W^0 \neq \emptyset$ and $-1 < p < 0$. Let $w_0 \in W^0$. Then clearly each of the functions

$$w_n(t) = w_{n-1}(t) + pw_{n-1}(t-\tau) \quad \text{for } n = 1, 2, \ldots \tag{25}$$

belongs to W^0. We will now show that (23) holds with $\mu = m_0$, where m_0 is the constant in Lemma 1(a). To this end, let $\lambda \in \Lambda^0(w_n)$ and set

$$\phi_\lambda(t) = w_n(t)e^{\lambda \int_0^t Q(s)\,ds}.$$

Then eventually $\dot{\phi}_\lambda(t) \leqq 0$ and so eventually $\phi_\lambda(t)$ is a decreasing function. Finally observe that

$$\begin{aligned} &\dot{w}_{n+1}(t) + (\lambda+\mu)Q(t)w_{n+1}(t) \\ &\quad = -Q(t)w_n(t-\sigma) + (\lambda+\mu)Q(t)w_n(t) + (\lambda+\mu)Q(t)pw_n(t-\tau) \\ &\quad = Q(t)\left[-\phi_\lambda(t-\sigma)e^{-\lambda\int_0^{t-\sigma}Q(s)\,ds} + (\lambda+\mu)\phi_\lambda(t)e^{-\lambda\int_0^t Q(s)\,ds}\right. \\ &\qquad\qquad \left. +(\lambda+\mu)p\phi_\lambda(t-\tau)e^{-\lambda\int_0^{t-\tau}Q(s)\,ds}\right] \\ &\quad \leqq Q(t)\phi_\lambda(t)e^{-\lambda\int_0^t Q(s)\,ds}(-e^{\lambda\sigma_1} + \lambda + \mu + \lambda pe^{\lambda\tau_1} + \mu pe^{\lambda\tau_1}). \end{aligned} \tag{26}$$

By using Lemma 1(a) and the fact that $p < 0$ we see that

$$\begin{aligned} \dot{w}_{n+1}(t) + (\lambda+m_0)Q(t)w_{n+1}(t) &\leqq Q(t)\phi_\lambda(t)e^{-\lambda\int_0^t Q(s)\,ds}(-m_0+m_0) \\ &= 0. \end{aligned}$$

This shows that $\lambda + m_0 \in \Lambda^0(w_{n+1})$ and the proof in Case I is complete.

Case II. $W^0 \neq \emptyset$ and $p > 0$. By Lemma 1(b) we know that $\sigma > \tau$. Here we will also use the sequence (25). Finally we claim that (23) holds with

$$\mu = m_0/1 + pe^{\lambda^*\tau_1},$$

where λ^* is the constant in Lemma 5(a). Indeed (26) implies that

$$\begin{aligned} &\dot{w}_{n+1}(t) + (\lambda+\mu)Q(t)w_{n+1}(t) \\ &\quad \leqq Q(t)\phi_\lambda(t-\tau)e^{-\lambda\int_0^t Q(s)\,ds}(-e^{\lambda\sigma_1} + \lambda + \mu + \lambda pe^{\lambda\tau_1} + \mu pe^{\lambda\tau_1}) \\ &\quad \leqq Q(t)\phi_\lambda(t-\tau)e^{-\lambda\int_0^t Q(s)\,ds}[-m_0 + \mu(1+pe^{\lambda\tau_1})] \\ &\quad \leqq Q(t)\phi_\lambda(t-\tau)e^{-\lambda\int_0^t Q(s)\,ds}[-m_0 + \mu(1+pe^{\lambda^*\tau_1})] \\ &\quad = 0. \end{aligned}$$

This shows that $\lambda + \mu \in \Lambda^0(w_{n+1})$ and the proof in Case II is complete.

Case III. $W^\infty \neq \varnothing$ and $\sigma \leqq \tau$. In this case we arrive at a contradiction by using the sequence

$$w_n(t) = -[w_{n-1}(t) + pw_{n-1}(t-\tau)] \quad \text{for } n = 1, 2, \ldots,$$

where w_0 is some fixed element of W^∞ and by taking

$$\mu = m_0/(-p-1).$$

Of course, $p < -1$ in this case. The proof is as in Cases I and II but here we utilize the substitution

$$\psi_\lambda(t) = e^{-\lambda \int_0^t Q(s)\,ds} w_n(t). \tag{27}$$

Case IV. $W^\infty \neq \varnothing$ and $\sigma > \tau$. In this case we use the sequence

$$w_n(t) = -[w_{n-1}(t) + pw_{n-1}(t-\tau)] + \int_{t-\sigma}^{t-\tau} Q(s) w_{n-1}(s)\,ds \quad \text{for } n = 1, 2, \ldots,$$

where w_0 is some fixed element of W^∞. We also take

$$\mu = \frac{m_0}{-p-1+(1/\lambda_0)} \quad \text{with} \quad \lambda_0 = \frac{1}{-p+\sigma_1-\tau_1}$$

which by Lemma 4(b) lies in $\wedge^\infty(w_n)$ for every $n = 1, 2, \ldots$. In this case we also use (27) with $\lambda \geqq \lambda_0$.

The proof of Theorem 1 is complete.

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