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PROPERTIES OF LOCALLY H-CLOSED SPACES

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ABSTRACT. This paper investigates the properties of locally H-closed spaces with regard to extensions, subspaces, and functions. We solve the H-closed extension remainder problem by showing that a space is locally H-closed if and only if it has a θ-closed remainder in some H-closed extension. In fact, an H-closed space is Urysohn iff every H-closed subspace is θ-closed. We solve the locally H-closed subspace problem by giving a necessary and sufficient condition for a subspace of a locally H-closed space to be locally H-closed. In particular, an open subspace of a locally H-closed space is locally H-closed if and only if its boundary is a θ-closed subspace of its closure. An H-closed space is shown to be compact if and only if every open subset is locally H-closed. A retract of a locally H-closed space is locally H-closed.

1. INTRODUCTION

Recall that a space is said to be *H-closed* iff it is a closed subspace of every Hausdorff embedding space. A space is *locally H-closed*, or LHC, iff every point has an H-closed nhood. Several open questions regarding LHC spaces will be solved in this paper.

In his 1970 study of LHC spaces, Porter noticed that the LHC property, unlike local compactness, is not hereditary on open subspaces. He also asked for a necessary and sufficient condition for a subspace of an LHC space to be LHC. In §3, we show that a subspace of an LHC space is LHC iff its closure less the subspace is a θ-closed subset of its closure, and its nowhere dense points are LHC. For open subsets, we have a much more concise answer. An open subset of an LHC space is LHC iff its boundary is a θ-closed subspace of its closure. We also show that an H-closed space is compact iff every open subset is LHC, in contrast to the famous theorem of M. H. Stone: "An H-closed space is compact iff every closed subspace is H-closed."

How do H-closed extensions of LHC spaces behave? In 1950, Obreanu showed that LHC spaces have one-point H-closed extensions, and also proved the existence of a projective maximum and a projective minimum for such ex-

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tensions. In 1973 and 1975, Porter and Votaw showed that a space is LHC iff its remainder in the Fomin extension is compact. In 1986, Tikoo investigated remainders of H-closed extensions, and, noticing that locally compact spaces always have compactifications with compact remainders, asked if locally H-closed spaces are characterized by having H-closed extensions with H-closed remainders. In §4, we show that the answer to Tikoo's question is negative, but, that a space is LHC iff any H-closed extension has a θ-closed remainder. In resolving this question, it was necessary to examine H-closed spaces in which every H-closed subspace is θ-closed. Such spaces were first considered by Velichko in 1965, and related questions have been recently addressed by Porter and Tikoo. We show that an H-closed space has the property that every H-closed subspace is θ-closed iff the space is Urysohn.

We study retract questions in §5, and show that a retract of an LHC space is LHC. In particular, the rationals are not a retract of any LHC space.

2. Preliminaries

Our primary source for terminology and results is the treatise by Porter and Woods [12]. Other results can be found in Engelking [3]. All spaces are Hausdorff, and all functions are continuous.

Dix H. Pettey has suggested that an open filterbase be called *quasi-free* iff none of its members has an H-closed closure. Then, we have the following filterbase characterization of LHC spaces.

2.1. **Proposition.** *A space is LHC iff every quasi-free filterbase is contained in some free open ultrafilter.*

Let X be a space, and B a subset. In [15], $x \in X$ is defined to be a *θ-limit point* of B iff every closed nhood of x intersects B. B is said to be *θ-closed* iff it contains all of its θ-limit points. It is easy to show that the intersection of any collection of θ-closed subspaces is θ-closed. Dikranjan and Giuli have show in [2] that the only subspaces that are θ-closed in all embedding spaces are the compact subspaces.

Let $f : X \to Y$, where X and Y are arbitrary spaces and f is not necessarily continuous. f is said to be *θ-continuous* iff for any $x \in X$ and any nhood V of $f(x)$, there exists a nhood U of x such that $f(\mathrm{Cl}_X U) \subseteq \mathrm{Cl}_Y V$.

A subset is said to be *regular open* iff it is the interior of its closure. A subset is *regular closed* iff its complement is regular open. A space is said to be *semiregular* iff it has a basis of regular open subsets. A space for which any two points have disjoint closed nhoods is a *Urysohn* space. A space for which there is no coarser Hausdorff topology is called *minimal Hausdorff.*

Every H-closed space has a coarser minimal Hausdorff topology. One of the classical problems in H-closed spaces is to show that the rationals do not contain a coarser minimal Hausdorff topology, and, although the original question has been answered by several authors, this line of inquiry continues today in studies of Katětov spaces such as [9]. A space is said to be a *Katětov space* if it has a coarser minimal Hausdorff topology. The search for an internal characterization of Katětov spaces is an open question. LHC spaces are Katětov spaces.

The next example of a noncompact, minimal Hausdorff space is presented by Herrlich in [4].

2.2. **Example.** Let X be the coarsest H-closed topology on the unit interval containing both the compact topology and the rationals. Let Y be the space resulting from identifying the free union of two copies of X at irrational points. Then Y is a noncompact minimal Hausdorff space, which can be thought of as two outer layers of rationals surrounding one inner layer of irrationals.

The H-closed equivalent of the Stone-Čech compactification, called the *Katětov extension*, and denoted by κX, can be realized for any Hausdorff space by fixing all the free open ultrafilters.

3. Subspaces

When is a subspace of an LHC space LHC? Porter makes the first assault on the problem in [7], when he shows that an open subset of even an H-closed space need not be LHC. So, we will characterize LHC open subspaces of H-closed spaces. He also shows that each LHC subspace of an arbitrary space is the intersection of an open set and a closed set, and observes that this is not a sufficient condition for a subspace to be LHC. We will improve upon this result, and give a partial converse. Porter asks for a necessary and sufficient condition for a subspace of an LHC space to be LHC. We will give such a condition, and show that, for some important special cases, the LHC subspace problem has a succinct answer.

Let us first consider the problem of determining when an open subspace of an H-closed space is LHC. If we recall that every space is open in its H-closed Katětov extension, we will perceive the need for a strong condition for an open subspace to be LHC.

3.1. **Lemma.** *Let M be a subspace of the space X.*

(1) *If $X \setminus M$ is θ-closed in X, then $\mathrm{Cl}_X M \setminus M$ is θ-closed in $\mathrm{Cl}_X M$.*
(2) *If M is open and $\mathrm{Cl}_X M \setminus M = \mathrm{Bd}_X M$ is θ-closed in $\mathrm{Cl}_X M$, then $X \setminus M$ is θ-closed in X.*
(3) *If M is LHC, then $\mathrm{Cl}_X M \setminus M$ is θ-closed in $\mathrm{Cl}_X M$.*

Proof. The proof is routine and is left to the reader.

We can now solve the question of when an open subspace of an H-closed space is LHC.

3.2. **Proposition.** *Let X be H-closed and B an open subset. B is LHC iff $X \setminus B$ is a θ-closed subset of X iff $\mathrm{Bd}\, B$ is a θ-closed subset of $\mathrm{Cl}_X B$.*

Proof. We will show that if B is LHC, then $X \setminus B$ is a θ-closed subset of X, and leave the rest of the proof to the reader. Let b be an element of the LHC space B. Since B is LHC, there exists some H-closed nhood of b contained in B, and this closed nhood is disjoint from $X\setminus B$, which is therefore θ-closed.

We conclude our study of LHC open subspaces of H-closed spaces with the next result, which contrasts with the famous theorem of M. H. Stone, who proved in [13] that an H-closed space is compact if and only if every closed subspace is H-closed. We say that a space is *rim θ-closed* iff it has a basis of open sets whose boundaries are θ-closed subsets of their closures. Recall that Pettey showed in [6] that there exists a noncompact example of an H-closed space that has a basis of open sets whose boundaries are H-closed.

3.3. **Theorem.** *An H-closed space X is compact iff it is rim θ-closed iff every open subset is LHC iff it has an open base of LHC sets.*

Proof. In view of Proposition 3.2, we only need show that an H-closed space is compact iff every basic open subset is LHC. Towards that goal, we note that one direction is easy, since a compact space has a basis of open sets with compact closures. So, we will assume that for some base, every basic open subset is LHC, and show that X is compact.

Let B be an open subset of X. Then, let $\{O_\alpha, \alpha \in A\}$ be a collection of basic open sets with boundaries that are θ-closed subsets of their closures, such that $B = \bigcup_{\alpha\in A} O_\alpha$. By Lemma 3.1, we know that $X\setminus O_\alpha$ is a θ-closed subset of X for each α, and the intersection of an arbitrary collection of θ-closed subsets is a θ-closed subset. Therefore, $X \setminus B = \bigcap_{\alpha\in A}\{X \setminus O_\alpha\}$ is a θ-closed subset of X. By Proposition 3.2, B is LHC, and every open subset of the H-closed space X is LHC; thus every closed subset is θ-closed, which is equivalent to saying that the space is regular. But a regular, H-closed space is compact. Thus, the proof is complete.

3.4. *Remark.* Let X be the space obtained from the compact unit interval by letting the sequence $\{1/n|n \text{ an integer}\}$ be closed. Then X is an example of a noncompact H-closed space in which every closed subset is LHC.

Having resolved the question of open LHC subspaces of H-closed spaces, we will now study the LHC subspace problem for LHC spaces.

3.5. **Example.** Example 2.2 is an LHC space in which the outer layers of rationals is an open subset that is not LHC, and the inner layer of irrationals is a closed subset which is not LHC.

We will tackle the problem of characterizing LHC subspaces of LHC spaces by working on specific cases first. We will first develop a useful lemma, and then consider a class of subspaces that includes the open subspaces.

3.6. **Lemma.** *Let X be LHC and B a subset. Then $\mathrm{Cl}_X B$ is LHC if there exists some H-closed extension of X, say hX, such that $\mathrm{Cl}_{hX} B$ is an H-closed subset of hX.*

Proof. Suppose that there exists some H-closed extension of X, say hX, such that $\mathrm{Cl}_{hX}B$ is an H-closed subset of hX. Then, if αX is the projective minimum one-point H-closed extension of X, there exists a continuous function $f : hX \to \alpha X$ such that $f(\mathrm{Cl}_{hX}B) = \mathrm{Cl}_{\alpha X}B$, and $\mathrm{Cl}_{\alpha X}B$ is the continuous image of an H-closed space and, therefore, must itself be H-closed. But $\mathrm{Cl}_{\alpha X}B$ is then $\mathrm{Cl}_X B$ or a one-point H-closed extension of $\mathrm{Cl}_X B$, so $\mathrm{Cl}_X B$ must be LHC.

3.7. **Theorem.** *Let X be LHC and B be a subset whose closure is LHC. B is LHC iff $\mathrm{Cl}_X B \setminus B$ is a θ-closed subset of $\mathrm{Cl}_X B$.*

Proof. First, suppose that B is LHC, and let $b \in B$. Then, there exists an H-closed nhood of b which is contained in B, which is disjoint from $\mathrm{Cl}_X B \setminus B$. Therefore, $\mathrm{Cl}_X B \setminus B$ is a θ-closed subset of $\mathrm{Cl}_X B$.

To get the other direction, let $b \in B$. Then, there is an open set M of the space $\mathrm{Cl}_X B$ such that $b \in M$ and the closure of M in $\mathrm{Cl}_X B$ is H-closed. There also exists an open set N of the space Cl_X B such that $b \in N$, and the closure of N in the space $\mathrm{Cl}_X B$ misses $\mathrm{Cl}_X B \setminus B$. Let $U = M \cap N$. Then $b \in U$, and the closure of U in $\mathrm{Cl}_X B$ is H-closed and contained in B. Thus, B is LHC.

The hypothesis in Theorem 3.7 includes what can be thought of as "trapped" subspaces, i.e., subspaces with nonempty interior that are contained in the closure of their interior. We can now answer the LHC subspace question for open subsets.

3.8. **Theorem.** *An open subset of an LHC space is LHC iff its boundary is a θ-closed subset of its closure iff its complement is θ-closed.*

Proof. Let X be LHC, and B any open subset of X. Since X is LHC, it is open in any H-closed extension, so B is also open in any H-closed extension of X, and its closure in that H-closed extension is H-closed. Thus, from Lemma 3.6, $\mathrm{Cl}_X B$ is LHC. Since B is open, $\mathrm{Cl}_X B \setminus B = \mathrm{Bd}\ B$. Then, our proposition follows immediately from Theorem 3.7.

Closed subsets of LHC spaces are not necessarily LHC. However, the most important class of closed subspaces does inherit the LHC property.

3.9. **Proposition.** *Let X be a LHC space. Then every regular closed subset is LHC.*

Proof. Let B be a regular closed subset of the LHC space X. Then $\mathrm{Int}_X B$ is nonempty. If αX is a one-point H-closed extension of X, then $\mathrm{Cl}_{\alpha X}\mathrm{Int}_X B$ is a one-point H-closed extension of B. Therefore, B is LHC.

We can now state the general solution to the LHC subspace problem. Recall that, for $B \subseteq X$, the *nowhere dense points* of B are the elements of B that are not elements of $\mathrm{Cl}_X \mathrm{Int}_X B$.

3.10. **Theorem.** *Let X be LHC and $B \subset X$. Then B is LHC iff:*

(1) *$\mathrm{Cl}_X B \setminus B$ is a θ-closed subset of $\mathrm{Cl}_X B$, and*
(2) *the set of nowhere dense points of B is LHC.*

Proof. Suppose X is LHC, and conditions (1) and (2) are met. B is the union of the set of its LHC nowhere dense points and $B \cap \mathrm{Cl}_X \mathrm{Int}_X B = \mathrm{Cl}_B \mathrm{Int}_X B$. But $\mathrm{Cl}_X \mathrm{Int}_X B$ is regular closed, thus LHC by Proposition 3.9, and an argument similar to that in the second part of Theorem 3.7 shows that $B \cap \mathrm{Cl}_X \mathrm{Int}_X B$ is LHC. Since B is the union of two LHC spaces, it must be LHC.

Now suppose B is LHC. First, we see that condition (1) follows from the third part of Lemma 3.1. Then, we complete the proof by noticing that the set of nowhere dense points, $B \setminus \mathrm{Int}_B \mathrm{Cl}_B \mathrm{Int}_X B$, is regular closed in the LHC subspace B, thus, from Proposition 3.9, is itself LHC, and condition (2) is satisfied.

3.11. *Remark.* Our proof of Theorem 3.10 also shows that, in a LHC space, a closed LHC subspace is the disjoint union of a regular closed LHC subset and a nowhere dense LHC subset. In [12], Porter and Woods use p-maps to show that in an H-closed space, an H-closed subspace is the union of a regular closed H-closed subset and a nowhere dense H-closed subset.

3.12. *Remark.* In [7], Porter shows that if B is a LHC subspace of the LHC space X, then $\mathrm{Cl}_X B \setminus B$ is closed. We have improved on this result by showing that, if B is LHC, then $\mathrm{Cl}_X B \setminus B$ is actually a θ-closed subset. In addition, we have established a partial converse by showing that if $\mathrm{Cl}_X B$ is LHC and $\mathrm{Cl}_X B \setminus B$ is a θ-closed subset of $\mathrm{Cl}_X B$, then B is LHC.

4. Extensions

The primary goal of this section is to characterize LH C spaces by their remainders in H-closed extensions. The first result follows immediately from our characterization of open LHC subspaces of H-closed spaces and the fact that an LHC space is open in every embedding subspace.

4.1. **Theorem.** *A space is LHC iff the remainder is θ-closed in every (some) H-closed extension. A dense subspace of an H-closed space is LHC iff its complement is θ-closed.*

4.2. *Remark.* There is a subtle stumbling block in the study of remainders of H-closed extensions. Locally compact spaces have closed remainders in their compactifications, and, due to the nature of compact spaces, local properties such as being closed become global properties such as compactness. For LHC spaces, we see that the distinguishing characteristic is a local property.

In [14], Tikoo asks if only LHC spaces have the property that each H-closed extension has an H-closed remainder. We answer this question by constructing a space that is not LHC but that has an H-closed extension with an H-closed remainder. We find an H-closed space which has a subspace that is H-closed but not θ-closed, and then manuever a bit, using a folklore technique, so that the subspace becomes nowhere dense in a larger H-closed space. We then revisit the idea of an H-closed space for which some H-closed subspace is not

θ-closed, but, for now, we assume that such examples exist (notice that Example 2.2 is such a space).

4.3. **Theorem.** *There exists a space that is not LHC and has an H-closed extension with H-closed remainder.*

Proof. Let X be an H-closed space such that $B \subset X$ is H-closed but not θ-closed. Let I be the compact unit interval. Then $X \times I$ is an H-closed extension of $X \times I \setminus B \times \{0\}$, and $B \times \{0\}$ is an H-closed remainder. Let $p \in X$ be a point such that every closed nhood of $\{p\}$ meets B. Such a point exists since B is not a θ-closed subset of X. Then, every closed nhood of $(p, 0)$ meets $B \times \{0\}$, which is therefore not θ-closed; hence, by Theorem 4.1, its complement is not LHC.

Thus, the answer to Tikoo's question is negative. But we have encountered the need to know more about H-closed spaces in which some subspace is H-closed but not θ-closed. Porter and Tikoo show in [9] that there exists a θ-closed subset of an H-closed space which is not H-closed, and there exists an H-closed subspace of an H-closed space which is not the θ-closure of any subset in the space. These questions and our own investigation prompted the following result, which extends a theorem of Velichko in [15].

4.4. **Proposition.** *Let X be an H-closed space. Every H-closed subspace is θ-closed iff X is Urysohn.*

Proof. Let x and y be any two points in X, and let N_x be any nhood of x whose closure misses y. Then Cl N_x is H-closed, thus θ-closed by hypothesis, so there exists some closed nhood N_y such that Cl $N_x \cap N_y = \varnothing$. Therefore, X is Urysohn.

The other direction follows from a result of Velichko in [15].

4.5. *Remark.* Porter [8] asks if there exists an noncompact minimal Hausdorff space in which every minimal Hausdorff subspace is θ-closed. Example 2.2 is such a space. The related question of whether or not there exists a noncompact H-closed space in which all H-closed subsets are minimal Hausdorff is open.

5. RETRACTS

Retracts of any space are closed, and local compactness is hereditary on closed subspaces, so it is easy to see that a retract of a locally compact space must be locally compact. The retract question for H-closed spaces is also easy. The continuous image of an H-closed space is H-closed, so a retract of an H-closed space must be H-closed. But closed LHC subspaces need not be LHC, and the continuous image of an LHC space need not be LHC. Therefore, the next result was unexpected.

5.1. **Theorem.** *A retract of a LHC space is LHC.*

Proof. Let B be a retract of the LHC space X, and let $f : X \to B$ be a continuous function that is the identity map on B. Let $b \in B$, and N_b be a nhood of b in B, i.e., there exists some M_b, open in X such that $N_b = M_b \cap B$. Since X is LHC, there exists some nhood O_b, open in X, such that $\mathrm{Cl}_X O_b$ is H-closed, and $O_b \subseteq M_b$. Notice that $O_b \cap B$ is a nhood of b in B. But $b = f(b) \in f(O_b \cap B) \subseteq f(\mathrm{Cl}_X O_b)$, and this last term is H-closed. But $f(O_b \cap B) = \mathrm{id}(O_b \cap B) = O_b \cap B$. Thus, $b \in O_b \cap B \subseteq f(\mathrm{Cl}_X O_b) \subseteq B$, and $\mathrm{Cl}_B(O_b \cap B)$ is H-closed. Since every element of B has an H-closed nhood in B, B is LHC.

5.2. *Remark.* The above result holds true even for θ-continuous functions. If X is a space and B is a subspace, we say that B is a *θ-retract* of X iff there exists a θ-continuous function $f : X \to B$ that agrees with the identity function on B. A slight variation of Theorem 5.1 shows that θ-retracts of LHC spaces are LHC.

5.3. **Corollary.** *The rationals are not a retract of any LHC space.*

5.4. *Remark.* Retracts of LHC spaces are LHC, thus Katětov. But, Porter and Tikoo show in [9] that Katětov spaces have a compact pre-image, and are the remainder of an H-closed extension of some discrete space. So, retracts of LHC spaces are particularly well behaved.

5.5. *Remark.* How can the retracts of an LHC space be characterized? Certainly, they are closed and LHC. But not every closed LHC subspace of an LHC space is a retract of the space. A finite subset of a connected Hausdorff space is closed and LHC, but if it contains more than one point, then it cannot be a retract of the space.

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