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A FRÉCHET-SCHWARTZ SPACE WITH BASIS HAVING A COMPLEMENTED SUBSPACE WITHOUT BASIS

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ABSTRACT. Using a method introduced by Pełczyński we show that for a nuclear Fréchet space E without basis we can find a Fréchet-Schwartz space F with basis containing a complemented, isomorphic copy of E.

We consider the famous problem of whether a complemented subspace of a Fréchet space with basis has a basis. For nuclear Fréchet spaces this question was asked by Pełczyński, and it still remains open; for Banach spaces the problem was solved by Szarek [S]. This paper contains the solution in the case of Fréchet-Schwartz spaces.

For nuclear and Schwartz spaces several partial positive results have been obtained by Mityagin, Henkin, Vogt, Dubinsky, Fachinger, and Krone (see [D1, D2, F, K, M1, M2, M3, V]).

Our proof is a combination of the fact that there exists a nuclear Fréchet space without a basis but having a suitable finite-dimensional decomposition, with the ideas of the elegant construction of Pełczyński [P] showing that each Fréchet space with the bounded approximation property is isomorphic to a complemented subspace of a Fréchet space having a basis. (The same strategy also leads to a solution in the case of Banach spaces; see [S].)

We now state the main result:

Theorem. *There exists a Fréchet-Schwartz space F with a basis having a complemented nuclear subspace E without basis.*

Proof. We shall use the method of [P]. For our purposes some nontrivial changes to [P] are necessary, so we give the whole construction in detail.

Recall ([J, §21.10]) that there exists a nuclear Fréchet space E without basis, the topology of which is determined by the norms

$$\|x\|_k := \sum_{n=1}^{\infty} n^{k-1} \|x_n\|_k^{(n)}, \tag{1}$$

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where $x = (x_n)_{n=1}^{\infty} \in E$ and $x_n \in \mathbf{K}^2$ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, the scalar field) and $\|\cdot\|_k^{(n)}$ is for all n, k a norm in $\mathbf{K}^2$ such that $\|\cdot\|_k^{(n)} \le \|\cdot\|_{k+1}^{(n)}$ for all n, k. The space E consists exactly of all sequences (x_n) such that (1) is finite for all k. For all n, let us denote by $M_n \cong \mathbf{K}^2$ the nth coordinate space of E. (We conjecture that it should be possible to carry out the following construction using any nuclear Fréchet space without a basis but with a finite-dimensional decomposition, instead of this special E.)

Let us denote by P_1 and P_2 the projections $P_j : x \mapsto (x|e_j)$, $j = 1, 2$, in $\mathbf{K}^2$ where (e_1, e_2) is the canonical basis of $\mathbf{K}^2$ and $(\cdot|\cdot)$ denotes the scalar product. We consider the projection also in each M_n and choose for all $n = 0, 1, 2, \dots$ the natural numbers $\rho(n)$ such that $\rho(0) = 0$, $\rho(n) \ge 1$ for $n \ge 1$, and

$$\|P_j x\|_k^{(n)} \le \rho(n) \|x\|_k^{(n)} \tag{2}$$

for $j = 1, 2$, all k, $1 \le k \le n$, and all $x \in M_n$. We define the operators $Q_i^{(n)} \in L(M_n)$, $i = 1, 2, \dots, 2\rho(n)$, by

$$Q_i^{(n)} := \rho(n)^{-1} P_j \tag{3}$$

if $i = 2t + j$ for some $t = 0, \dots, \rho(n) - 1$ and $j = 1, 2$. We have then, for all $x \in M_n$,

$$\sum_{i=1}^{2\rho(n)} Q_i^{(n)} x = \sum_{i=1}^{\rho(n)} \rho(n)^{-1} P_1(x) + \rho(n)^{-1} P_2(x) = x, \tag{4}$$

and for all k, $1 \le k \le n$, for all q, $1 \le q \le 2\rho(n)$, by (2),

$$\begin{aligned} \left\| \sum_{i=1}^{q} Q_i^{(n)} \right\|_{L(M_n, k)} &\le \left\| \sum_{i=1}^{q^*} Q_i^{(n)} \right\|_{L(M_n, k)} + \|Q_q^{(n)}\|_{L(M_n, k)} \\ &= \|2^{-1} q^* \rho(n)^{-1} id_{M_n}\|_{L(M_n, k)} + 1 \le 2, \end{aligned}$$

where we have denoted by $\|\cdot\|_{L(M_n, k)}$ the operator norm with respect to the Banach space $(M_n, \|\cdot\|_k^{(n)})$ and by q^* the largest even number not greater than q. As a consequence, we see that for all k there exists a $C_k > 0$ such that for all $n \in \mathbf{N}$ and for all q, $1 \le q \le 2\rho(n)$,

$$\left\| \sum_{i=1}^{q} Q_i^{(n)} \right\|_{L(M_n, k)} \le C_k. \tag{5}$$

For all $n \in \mathbf{N}$, $1 \le i \le 2\rho(n)$, we now define the 1-dimensional subspace $M_{n,i}$ of E by $Q_i^{(n)} \widehat{Q}_n(E)$, where $\widehat{Q}_n$ is the canonical projection from E onto M_n.

We define F as the space

$$F := \{(x_{n,i})_{n \in \mathbf{N}, 1 \le i \le 2\rho(n)} | x_{n,i} \in M_{n,i}, \ q_k((x_{n,i})) < \infty\}, \tag{6}$$

where the norms q_k, $k \in \mathbf{N}$ in F are defined by

$$q_k((x_{n,i})) = \sum_{n=1}^{\infty} \sup_{1 \le m \le 2\rho(n)} \left\| \sum_{i=1}^{m} x_{n,i} \right\|_k . \tag{7}$$

Clearly $q_k \le q_{k+1}$ for all k. We leave it to the reader to verify that F becomes a Fréchet space when endowed with the norms (7). Note that if $(x_{n,i}) \in F$, then the sum

$$\sum_{n \in \mathbf{N}} y_n , \tag{8}$$

where $y_n = \sum_{i=1}^{2\rho(n)} x_{n,i}$, converges in E, as easily seen from the definition of F. It is also clear from definitions that, for all $x \in E$,

$$\sum_{n=1}^{\infty} \sum_{i=1}^{2\rho(n)} Q_i^{(n)} \widehat{Q}_n x = x , \tag{9}$$

where the convergence is as in (8).

We now prove the essential statements.

1. *The space F admits a basis.* Indeed, we pick a nonzero element $x_{n,i}$ in each $M_{n,i}$. Let for all $m \in \mathbf{N}$ the element y_m be equal to $x_{n,i}$, where $m = 2\rho(0) + 2\rho(1) + \cdots + 2\rho(n-1) + i (1 \le i \le 2\rho(n))$. It is an immediate consequence of the definition of F that the linear combinations of the elements y_m are dense in F. On the other hand it is easy to see, using the definitions, that , for all m and k and for all sequences of scalars $(a_t)_{t=1}^{\infty}$,

$$q_k\left(\sum_{t=1}^{m} a_t y_t\right) \le q_k\left(\sum_{t=1}^{m+1} a_t y_t\right) . \tag{10}$$

This implies by [J, 14.3.6] that (y_m) is a basis.

2. *The space E is isomorphic to a complemented subspace of F.* We define the mapping $T\colon E \to F$ by $Tx = (Q_i^{(n)} \widehat{Q}_n x)_{n \in \mathbf{N}, 1 \le i \le 2\rho(n)}$. Then by (5) and (9) we get, for $x \in E$,

$$\begin{aligned} \|x\|_k &= \left\| \sum_{n=1}^{\infty} \sum_{i=1}^{2\rho(n)} Q_i^{(n)} \widehat{Q}_n x \right\|_k = \sum_{n=1}^{\infty} \left\| \sum_{i=1}^{2\rho(n)} Q_i^{(n)} \widehat{Q}_n x \right\|_k \\ &\le \sum_{n \in \mathbf{N}} \sup_{m \le 2\rho(n)} \left\| \sum_{i=1}^{m} Q_i^{(n)} \widehat{Q}_n x \right\|_k \\ &= q_k(Tx) \le \sum_{n \in \mathbf{N}} C_k \|\widehat{Q}_n x\|_k = C_k \|x\|_k . \end{aligned} \tag{11}$$

This shows that T is an isomorphic embedding. The operator P, defined by

$$P((x_{n,i})_{n,i}) = \left(Q_i^{(n)} \widehat{Q}_n \left(\sum_{n'=1}^{\infty} \sum_{i'=1}^{2\rho(n)} x_{n',i'} \right) \right)_{n,i} , \tag{12}$$

is a projection from F onto $T(E)$; the fact that P is the identity on $T(E)$ is a consequence of (9). The continuity of P follows from

$$
\begin{aligned}
&q_k\left(\left(Q_i^{(n)}\hat{Q}_n\left(\sum_{n'}\sum_{i'} x_{n',i'}\right)\right)_{n,i}\right) \\
&= \sum_n \sup_{m\le 2\rho(n)} \left\| \sum_{i=1}^m Q_i^{(n)}\hat{Q}_n\left(\sum_{n'}\sum_{i'} x_{n',i'}\right)\right\|_k \\
(13)\qquad &= \sum_n \sup_{m\le 2\rho(n)} \left\| \sum_{i=1}^m Q_i^{(n)}\left(\sum_{i'=1}^{2\rho(n)} x_{n,i'}\right)\right\|_k \\
&\le \sum_n \sup_m \left\|\sum_{i=1}^m Q_i^{(n)}\right\|_{L(M_n,k)} \left\|\sum_{i=1}^{2\rho(n)} x_{n,i}\right\|_k \\
&\le \sum_n C_k \left\|\sum_{i=1}^{2\rho(n)} x_{n,i}\right\|_k \le C_k q_k((x_{n,i})).
\end{aligned}
$$

3. *The space F is Schwartz.* To prove this statement, we denote by N_{n_0} the subspace of F spanned by the elements $(x_{n,i})$ for which only the coordinates $x_{n_0,i}$, $1 \le i \le 2\rho(n_0)$ are nonzero. Let S_n be the natural projection from F onto N_n. Note that if $(x_{n,i}) \in N_{n_0}$, then

$$(14)\qquad \sum_{n,i} x_{n,i} \in M_{n_0} \subset E.$$

We can write

$$(15)\qquad q_k(x) = \sum_{n=1}^{\infty} q_k(S_n x).$$

Moreover, it follows from the definition of the norms $\|\cdot\|_k$ in (1) and (14) that, for $n \ge k+1$ and all $x \in N_n \subset F$ and for all $k \in \mathbf{N}$,

$$(16)\qquad \frac{q_k(x)}{q_{k+1}(x)} \le \frac{n^{k-1}}{n^k}.$$

Now a standard argument shows that F is a Schwartz space: if $k \in \mathbf{N}$, $\varepsilon > 0$ are given, we choose $n_0 > k$ such that $n^{k-1}/n^k < \varepsilon/2$ for $n > n_0$; then (15) and (16) imply that

$$U_{k+1} \cap \oplus_{n>n_0} N_n \subset \frac{\varepsilon}{2} U_k,$$

where U_k is the closed unit ball of q_k. The fact that each N_n, $n \le n_0$, is finite-dimensional then implies the existence of a finite number of elements $a_i \in F$ with

$$U_{k+1} \subset \bigcup_i (a_i + \varepsilon U_k).$$

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