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SOME REMARKS ABOUT ROSEN'S FUNCTIONS

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ABSTRACT. The main result is: Each Baire 2 function $f: I \to R$ whose set of continuity points is dense is the pointwise limit of a sequence of Darboux Baire $\frac{1}{2}$ functions.

Let $I = [0, 1]$ and R be the set of all reals. A function $f: I \to R$ is said to be Baire $\frac{1}{2}$ [6] if preimages of open sets are G_s-sets. (Rosen states these functions as Baire -.5 [6].) In [6] H. Rosen shows the following theorem:

Theorem 0. *Suppose $f: I \to R$ is a Darboux Baire $\frac{1}{2}$ function, and let D denote the set of points at which f is continuous. Then the graph of f/D is bilaterally c-dense in the graph of f.*

Remark 1. A function $f: I \to R$ is a Baire $\frac{1}{2}$ function iff it is ambiguously a Baire 1 function, i.e. preimages of open sets are G_s- and F_σ-sets simultaneously. Indeed if f is Baire $\frac{1}{2}$, then every open set U is the sum of closed sets F_n $(n = 1, 2, \dots)$, hence

$$f^{-1}(U) = f^{-1}\left(\bigcup_{n=1}^{\infty} F_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(F_n)$$

and $\bigcup_{n=1}^{\infty} f^{-1}(F_n)$ is an F_σ-set.

Remark 2. A function $f: I \to R$ is said to be a.e. continuous [4] if it is approximately continuous and continuous almost everywhere (in the sense of the Lebesgue measure). There is an a.e. continuous function $f: I \to R$ which is not Baire $\frac{1}{2}$.

Example 1. Indeed, let $P \subset I$ be a Cantor set of measure zero and let $A \subset P$ be a countable set such that $\mathrm{Cl}\,A = P$ ($\mathrm{Cl}\,A$ denotes the closure of the set A). Let $(a_n)_n$ be a sequence of all points of the set A. There is a family of closed intervals $J_{nm} \subset I - P$ $(n, m = 1, 2, \dots)$ such that:

(1) $J_{nm} \cap J_{rs} = \varnothing$ if $(n, m) \neq (r, s)$, $n, m, r, s = 1, 2, \dots$;
(2) a_n is a density point of the set $\bigcup_{m=1}^{\infty} J_{nm}$, $n = 1, 2, \dots$;

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(3) $\mathrm{Cl}(\bigcup_{m=1}^{\infty} J_{nm}) = \bigcup_{m=1}^{\infty} J_{nm} \cup \{a_n\}$, $n = 1, 2, \dots$;

(4) if $x \in I - P$, then there is an open neighborhood U of x such that the set of pairs (n, m) with $J_{nm} \cap U \neq \emptyset$ is empty or contains only one element.

There are also closed intervals $I_{nm} \subset \mathrm{Int}\, J_{nm}$ ($\mathrm{Int}\, J_{nm}$ denotes the interior of the set J_{nm}), $n, m = 1, 2, \dots$, such that a_n is a density point of the set $\bigcup_{m=1}^{\infty} I_{nm}$, $n = 1, 2, \dots$. Define, for $n = 1, 2, \dots$,

$$f_n(x) = \begin{cases} 2^{-n} & \text{for } x = a_n \text{ or } x \in I_{nm},\ m = 1, 2, \dots \\ 0 & \text{for } x \in I - \bigcup_{m=1}^{\infty} \mathrm{Int}\, J_{nm} - \{a_n\} \\ & \text{linear in the component intervals of the set } J_{nm} - \mathrm{Int}\, I_{nm} \\ & (m = 1, 2, \dots). \end{cases}$$

Obviously every function f_n $(n = 1, 2, \dots)$ is a.e. continuous everywhere and continuous at each point $x \neq a_n$. So the function

$$f = \sum_{n=1}^{\infty} f_n$$

is a.e. continuous everywhere and continuous at each point $x \neq a_n$, $n = 1, 2, \dots$. Because the set of all points a_n, $n = 1, 2, \dots$, is not a G_s-set, the set $\{x; f(x) > 0\}$ is not a G_s-set and f is not Baire $\frac{1}{2}$.

Remark 3. In [5] Preiss showed that every Baire 2 function $f: I \to R$ is the limit of a sequence of approximately continuous functions. From Theorem 0 it follows that every Baire $\frac{1}{2}$ function is quasicontinuous, i.e. for every $x \in I$, for every $r > 0$ and for every open neighborhood U of x there is a nonempty open set $V \subset U$ such that

$$|f(u) - f(x)| < r \quad \text{for all } u \in V.$$

(See [1].) So if $f: I \to R$ is the limit of a sequence of Baire $\frac{1}{2}$ functions then f is a Baire 2 function having dense the set of its continuity points [1].

Our main result is the following

Theorem 1. *Each Baire* 2 *function* $f: I \to R$ *whose set of continuity points is dense is the pointwise limit of a sequence of Darboux Baire* $\frac{1}{2}$ *functions.*

Proof. Denote by $C(f)$ the set of all continuity points of f. The set $C(f)$ is a dense G_s-set. There is a Baire 1 function $g: I \to R$ such that $g(x) = f(x)$ for every point $x \in C(f)$ [2, p. 342]. Let $h = f - g$. The set $C(h)$ of continuity points of h is dense and the level set $h^{-1}(0)$ is also dense. Since h is a Baire 2 function, there is a sequence of Baire 1 functions $k_n: I \to R$ such that every set $k_n(I)$ $(n = 1, 2, \dots)$ is finite and $h = \lim_{n\to\infty} k_n$ [2, pp. 294–295]. For $n = 1, 2, \dots$ denote by A_n the set of all points $x \in R$ at which $\mathrm{osc}\, h(x) \geq 4^{-n}$ and put

$$l_n(x) = \begin{cases} 0 & \text{for } x \in I - A_n, \\ k_n(x) & \text{for } x \in A_n. \end{cases}$$

We may assume without loss of generality that $A_{n+1} - A_n \neq \emptyset$ $(n = 1, 2, \dots)$ and $|k_n(x)| < 2^{-n}$ for $x \in I - A_n$. Evidently $h = \lim_{n\to\infty} l_n$ and l_n is continuous at every $x \in I - A_n$ $(n = 1, 2, \dots)$. There is a family of closed intervals K_{1kl} $(k = 1, 2, \dots$ and $l \leq k)$ such that:

(1) $K_{1kl} \subset I - A_{1+k+l}$, $k = 1, 2, \dots$ and $l \leq k$;
(2) $K_{1kl} \cap K_{1rs} = \emptyset$ for $(k, l) \neq (r, s)$, $k, r = 1, 2, \dots$ and $l \leq k$, $s \leq r$;
(3) for every $x \in A_1$ and for every closed interval $U \ni x$ and for every l there is $k \geq l$ such that $K_{1kl} \subset \operatorname{Int} U$;
(4) for every $x \in I - A_1$ there is an open neighborhood V of x such that the set of pairs (k, l) with $K_{1kl} \cap V \neq \emptyset$ is empty or contains only one element.

For the construction of such a family it suffices to choose in every complementary interval (a_n, b_n) of the set A_1 two sequences of closed intervals $[c_{1nk}, d_{1nk}]$ and $[c_{2nk}, d_{2nk}]$ such that:

(1) $c_{2n1} > d_{1n1} > c_{1n1} > d_{1n2} > c_{1n2} > \cdots \to a_n$;
(2) $c_{2n1} < d_{2n1} < c_{2n2} < d_{2n2} < \cdots \to b_n$;
(3) $[c_{ink}, d_{ink}] \cap A_{4^{n+k}} = \emptyset$, $k = 1, 2, \dots$ and $i = 1, 2$.

(See also [8, Lemma 2.1].) For every $k = 1, 2, \dots$ we define a continuous function $l_{1k}\colon K_{1k1} \to [-k, k]$ such that $l_{1k}(K_{1k1}) = [-k, k]$ and $l_{1k}(x) = 0$ if x is an endpoint of the interval K_{1k1}.

Let

$$h_1(x) = \begin{cases} l_{1k}(x) & \text{if } x \in K_{1k1},\ k = 1, 2, \dots \\ l_1(x) & \text{in the remaining case.} \end{cases}$$

Evidently h_1 is a center Darboux Baire $\frac{1}{2}$ function, because it is continuous at every point $x \in I - A_1$ and h_1/A_1 is a Baire 1 function such that the set $h_1(A_1)$ is finite.

In the second step we choose a family K_{2kl} $(k = 1, 2, \dots$ and $l \leq k)$ of closed intervals such that:

(1) $K_{2kl} \subset I - A_{2+k+l}$, $k = 1, 2, \dots$ and $l \leq k$;
(2) $K_{2kl} \cap K_{2rs} = \emptyset$ if $(k, l) \neq (r, s)$, $k, r = 1, 2, \dots$ and $l \leq k$ and $s \leq r$;
(3) $K_{2kl} \cap K_{1rs} = \emptyset$, $k, r = 1, 2, \dots$ and $l \leq k$ and $s \leq r$;
(4) for every $x \in \operatorname{Cl}(A_2 - A_1)$ and for every closed interval $V \ni x$ and for every l there is $k \geq l$ such that $K_{2kl} \subset \operatorname{Int} V$;
(5) for every $x \in I - A_2$ there is an open set $U \ni x$ such that the set of couples (k, l) with $K_{2kl} \cap U \neq \emptyset$ is empty or it contains only one element.

For every $k = 2, 3, \dots$, we define continuous functions $l_{21k}\colon K_{1k2} \to [-k, k]$ and $l_{22k}\colon K_{2k1} \to [-2^{-1}, 2^{-1}]$ such that $l_{21k}(K_{1k2}) = [-k, k]$, $l_{22k}(K_{2k1}) = [-2^{-1}, 2^{-1}]$, $l_{21k}(x) = 0$ for x an endpoint of K_{1k2} and $l_{22k}(x) = 0$ for x

an endpoint of K_{2k1}. Let

$$h_2(x) = \begin{cases} l_{21k}(x) & \text{for } x \in K_{1k2}, \ k = 2, 3, \dots \\ l_{22k}(x) & \text{for } x \in K_{2k1}, \ k = 2, 3, \dots \\ l_2(x) & \text{in the remaining case.} \end{cases}$$

The function h_2 is continuous at each point $x \in A_2$ and it is a Darboux Baire $\frac{1}{2}$ function.

Generally, in step $n > 2$ we remark that $K_{1kl} \cap A_n = \varnothing$ for $i < n$, $k = 1, 2, \dots, l \le k$ with $k + l \ge n$ and we construct a family of closed intervals K_{nkl}, $k = 1, 2, \dots$ and $l \le k$ such that:

(1) $K_{nkl} \subset I - A_{n+k+l}$, $k = 1, 2, \dots$ and $l \le k$;
(2) $K_{nkl} \cap K_{irs} = \varnothing$ if $i < n$ and $i + r + s \ge n$; $(k, r = 1, 2, \dots, l \le k, s \le r)$;
(3) $K_{nkl} \cap K_{nrs} = \varnothing$ if $(k, l) \ne (r, s)$, $k, r = 1, 2, \dots, l \le k$, $s \le r$;
(4) for every $x \in \mathrm{Cl}(A_n - A_{n-1})$, for every closed interval $V \ni x$ and for every l there exists $k \ge l$ such that $K_{nkl} \subset \mathrm{Int}\, V$;
(5) for every $x \in I - A_n$ there is an open set $U \ni x$ such that the set $\{(k, l);\, K_{nkl} \cap U \ne \varnothing\}$ is empty or it contains only one element.

For every pair (i, k) such that $i + k > n$ and $i \le n$ we define a continuous function $l_{nik}: K_{i,k,n-i+1} \to R$ such that $l_{nik}(x) = 0$ for x being an endpoint of K_{ink}, $l_{1nk}(K_{1kn}) = [-k, k]$ and $l_{nik}(K_{i,k,n-i+1}) = [-2^{-i+1}, 2^{-i+1}]$ for $i > 1$.

Let

$$h_n(x) = \begin{cases} l_{nik}(x) & \text{for } x \in K_{i,k,n-i+1}, \ i \le n, \ i + k > n \\ l_n(x) & \text{in the remaining case.} \end{cases}$$

h_n is a Darboux Baire $\frac{1}{2}$ function, because it is continuous at every point $x \in I - A_n$ and h_n/A_n is a Baire 1 function such that the set $h_n(A_n)$ is finite and $h_n(A_k - A_{k-1}) \subset (-2^{-k+1}, 2^{-k+1})$ for $k = 2, 3, \dots, n$. Obviously

$$\lim_{n\to\infty} h_n(x) = h(x) \quad \text{for } x \in \bigcup_{n=1}^{\infty} A_n \text{ and for } x \in I - \bigcup_{n=1}^{\infty} A_n - \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{l \le k} K_{nkl}.$$

Let $x \in K_{nkl}$ for some indices n, k, l. We have $h(x) = 0$. Let $r > 0$. There is an index m such that $2^{-m+1} < r$. For $i = 1, \dots, m$ let $n(i)$ denote 1 if $x \in I - \bigcup_{k=1}^{\infty} \bigcup_{l \le k} K_{ikl}$ or the index $l(x)$ such that $x \in K_{i,k(x),l(x)}$ in the remaining case. Let

$$N = 3 \max(n(1), n(2), \dots, n(m), m).$$

Observe that for $n > N$, $x \notin K_{i,k,n-i+1}$ for $i = 1, \dots, m$ and $i + k \ge n$ and

$$|h_n(x)| < 2^{-m+1} < r.$$

So $h = \lim_{n\to\infty} h_n$.

There is a sequence of continuous functions $g_n: I \to R$ $(n = 1, 2, \dots)$ such that $g = \lim_{n\to\infty} g_n$. Since g_n is continuous everywhere and h_n is a center

Darboux Baire 1 function and $h_n(A_n)$ is finite, so $f_n = g_n + h_n$ is a Darboux Baire $\frac{1}{2}$ function. We have also

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} g_n + \lim_{n\to\infty} h_n = g + h = f$$

and the proof is complete.

Remark 4. Suppose that $f: I \to R$ is a Darboux Baire $\frac{1}{2}$ function and $B \subset I$ is a dense set in I. From Theorem 0 it follows that for every $x \in I$ we have

$$\liminf_{\substack{t\to x \\ t\in B}} f(t) \le f(x) \le \limsup_{\substack{t\to x \\ t\in B}} f(t).$$

From [3] we derive the following theorem:

Theorem 2. *Suppose that $F: I^2 \to R$ is a function such that all the sections $F_x(t) = F(x, t)$ $(x, t \in I)$ are of Darboux Baire $\frac{1}{2}$ class (and continuous almost everywhere). If all the sections $F^y(t) = F(t, y)$ $(t, y \in I)$ have the Baire property (are measurable), then F has also the Baire property (is measurable).*

Example 2. There is a nonmeasurable function $F: I^2 \to R$ such that all the sections F_x, F^y are Darboux Baire $\frac{1}{2}$. Let $A \subset I$ be a Cantor set of positive measure and let $B \subset I^2$ be a nonmeasurable set of full exterior measure such that all the sections $B_x = \{t \in I; (x, t) \in B\}$ and $B^y = \{t \in I; (t, y) \in B\}$ are empty or contain one or two points [7]. In every complementary interval (a_n, b_n) of the set A we choose a closed interval $I_n \subset (a_n, b_n)$. For every $n = 1, 2, \ldots$ we define a continuous function $f_n: I_n \to R$ such that $f_n(I_n) = [0, 1]$ and $f_n(x) = 0$ if x is an endpoint of I_n.

Define for $(x, y) \in (A \times (R - A)) \cup ((R - A) \times A)$,

$$g(x, y) = \begin{cases} f_n(y) & \text{for } x \in A \text{ and } y \in I_n,\ n = 1, 2, \ldots, \\ f_n(x) & \text{for } y \in A \text{ and } x \in I_n,\ n = 1, 2, \ldots, \\ 0 & \text{in the remaining case.} \end{cases}$$

Since the set $(A \times (R - A)) \cup ((R - A) \times A)$ is closed in the space $H = (I \times I) - (A \times A)$, so there exists a continuous function $h: H \to [0, 1]$ such that $h(x, y) = g(x, y)$ for $(x, y) \in (A \times (R - A)) \cup ((R - A) \times A)$.

Let

$$F(x, y) = \begin{cases} h(x, y) & \text{for } (x, y) \in H, \\ 1 & \text{for } (x, y) \in (A \times A) \cap B, \\ 0 & \text{in the remaining cases.} \end{cases}$$

Since the set $(A \times A) \cap B$ is not measurable, so F is nonmeasurable. Evidently all the sections F_x, F^y are in the Darboux Baire $\frac{1}{2}$ class.

References

1. Z. Grande, *Sur la quasi-continuité et la quasi-continuité approximative*, Fund. Math. **129** (1988), 167–172.
2. C. Kuratowski, *Topologie* I, PWN Warszawa, 1958.

3. E. Marczewski and Cz. Ryll-Nardzewski, *Sur la mesurabilité des fonctions de plusieurs variables*, Annales Soc. Polon. Math. **25** (1953), 145–154.
4. R. J. O'Malley, *Approximately differentiable functions. The r-topology*, Pacific J. Math. **72** (1977), 207–222.
5. D. Preiss, *Limits of approximately continuous functions*, Czechoslovak Math. J. **96** (1971), 371–372.
6. H. Rosen, *Darboux Baire - .5 function*, Proc. Amer. Math. Soc. **110** (1990), 285–286.
7. W. Sierpiński, *Sur un problème concernant les ensembles mesurables superficiellement*, Fund. Math. **1** (1920), 112–115.
8. E. Strońska, *Algebry generowane przez pewne rodziny funkcji quasi ciągłych*, Doctoral thesis, Łódź University, 1990.

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