SOME REMARKS ABOUT ROSEN'S FUNCTIONS

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(Communicated by Andrew M. Bruckner)

Abstract. The main result is: Each Baire 2 function $f: I \rightarrow R$ whose set of continuity points is dense is the pointwise limit of a sequence of Darboux Baire $\frac{1}{2}$ functions.

Let $I = [0, 1]$ and $R$ be the set of all reals. A function $f: I \rightarrow R$ is said to be Baire $\frac{1}{2}$ [6] if preimages of open sets are $G$-sets. (Rosen states these functions as Baire -5 [6].) In [6] H. Rosen shows the following theorem:

Theorem 0. Suppose $f: I \rightarrow R$ is a Darboux Baire $\frac{1}{2}$ function, and let $D$ denote the set of points at which $f$ is continuous. Then the graph of $f/D$ is bilaterally c-dense in the graph of $f$.

Remark 1. A function $f: I \rightarrow R$ is a Baire $\frac{1}{2}$ function iff it is ambiguously a Baire 1 function, i.e. preimages of open sets are $G$- and $F$-sets simultaneously. Indeed if $f$ is Baire $\frac{1}{2}$, then every open set $U$ is the sum of closed sets $F_n$ $(n = 1, 2, \ldots)$, hence

$$f^{-1}(U) = f^{-1}\left(\bigcup_{n=1}^{\infty} F_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(F_n)$$

and $\bigcup_{n=1}^{\infty} f^{-1}(F_n)$ is an $F$-set.

Remark 2. A function $f: I \rightarrow R$ is said to be a.e. continuous [4] if it is approximately continuous and continuous almost everywhere (in the sense of the Lebesgue measure). There is an a.e. continuous function $f: I \rightarrow R$ which is not Baire $\frac{1}{2}$.

Example 1. Indeed, let $P \subset I$ be a Cantor set of measure zero and let $A \subset P$ be a countable set such that $\text{Cl} A = P$ (Cl $A$ denotes the closure of the set $A$). Let $(a_n)_n$ be a sequence of all points of the set $A$. There is a family of closed intervals $J_{nm} \subset I - P$ $(n, m = 1, 2, \ldots)$ such that:

1. $J_{nm} \cap J_{rs} = \emptyset$ if $(n, m) \neq (r, s), n, m, r, s = 1, 2, \ldots$;
2. $a_n$ is a density point of the set $\bigcup_{m=1}^{\infty} J_{nm}, n = 1, 2, \ldots$;

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(3) \( \text{Cl}(\bigcup_{m=1}^{\infty} J_{nm}) = \bigcup_{m=1}^{\infty} J_{nm} \cup \{a_n\}, \quad n = 1, 2, \ldots; \)

(4) if \( x \in I - P \), then there is an open neighborhood \( U \) of \( x \) such that the set of pairs \( (n, m) \) with \( J_{nm} \cap U \neq \emptyset \) is empty or contains only one element.

There are also closed intervals \( J_{nm} \subset \text{Int} J_{nm} \) (\( \text{Int} J_{nm} \) denotes the interior of the set \( J_{nm} \)), \( n, m = 1, 2, \ldots \), such that \( a_n \) is a density point of the set \( \bigcup_{m=1}^{\infty} I_{nm}, \quad n = 1, 2, \ldots \). Define, for \( n = 1, 2, \ldots \),

\[
 f_n(x) = \begin{cases} 
 2^{-n} & \text{for } x = a_n \text{ or } x \in I_{nm}, \quad m = 1, 2, \ldots, \\
 0 & \text{for } x \in I - \bigcup_{m=1}^{\infty} \text{Int} J_{nm} - \{a_n\} 
\end{cases}
\]

linear in the component intervals of the set \( J_{nm} - \text{Int} I_{nm} \) \( m = 1, 2, \ldots \).

Obviously every function \( f_n \) \( (n = 1, 2, \ldots) \) is a.e. continuous everywhere and continuous at each point \( x \neq a_n \). So the function

\[
 f = \sum_{n=1}^{\infty} f_n
\]

is a.e. continuous everywhere and continuous at each point \( x \neq a_n \), \( n = 1, 2, \ldots \). Because the set of all points \( a_n, \quad n = 1, 2, \ldots \), is not a \( G_\delta \)-set, the set \( \{x; f(x) > 0\} \) is not a \( G_\delta \)-set and \( f \) is not Baire \( \frac{1}{2} \).

Remark 3. In [5] Preiss showed that every Baire 2 function \( f: I \to R \) is the limit of a sequence of approximately continuous functions. From Theorem 0 it follows that every Baire \( \frac{1}{2} \) function is quasicontinuous, i.e. for every \( x \in I \), for every \( r > 0 \) and for every open neighborhood \( U \) of \( x \) there is a nonempty open set \( V \subset U \) such that

\[
|f(u) - f(x)| < r \quad \text{for all } u \in V.
\]

(See [1].) So if \( f: I \to R \) is the limit of a sequence of Baire \( \frac{1}{2} \) functions then \( f \) is a Baire 2 function having dense the set of its continuity points [1].

Our main result is the following

**Theorem 1.** Each Baire 2 function \( f: I \to R \) whose set of continuity points is dense is the pointwise limit of a sequence of Darboux Baire \( \frac{1}{2} \) functions.

**Proof.** Denote by \( C(f) \) the set of all continuity points of \( f \). The set \( C(f) \) is a dense \( G_\delta \)-set. There is a Baire 1 function \( g: I \to R \) such that \( g(x) = f(x) \) for every point \( x \in C(f) \) [2, p. 342]. Let \( h = f - g \). The set \( C(h) \) of continuity points of \( h \) is dense and the level set \( h^{-1}(0) \) is also dense. Since \( h \) is a Baire 2 function, there is a sequence of Baire 1 functions \( k_n: I \to R \) such that every set \( k_n(I) \) \( (n = 1, 2, \ldots) \) is finite and \( h = \lim_{n \to \infty} k_n \) [2, pp. 294–295]. For \( n = 1, 2, \ldots \) denote by \( A_n \) the set of all points \( x \in R \) at which \( \text{osc} h(x) \geq 4^{-n} \) and put

\[
 l_n(x) = \begin{cases} 
 0 & \text{for } x \in I - A_n, \\
 k_n(x) & \text{for } x \in A_n.
\end{cases}
\]
We may assume without loss of generality that $A_{n+1} - A_n \neq \emptyset$ $(n = 1, 2, \ldots)$ and $|k_n(x)| < 2^{-n}$ for $x \in I - A_n$. Evidently $h = \lim_{n \to \infty} l_n$ and $l_n$ is continuous at every $x \in I - A_n$ $(n = 1, 2, \ldots)$. There is a family of closed intervals $K_{1kl}$ $(k = 1, 2, \ldots$ and $l \leq k)$ such that:

1. $K_{1kl} \subset I - A_{1+k+l}$, $k = 1, 2, \ldots$ and $l \leq k$;
2. $K_{1kl} \cap K_{1rs} = \emptyset$ for $(k, l) \neq (r, s)$, $k, r = 1, 2, \ldots$ and $l \leq k$, $s \leq r$;
3. for every $x \in A_1$ and for every closed interval $U \ni x$ and for every $l$ there is $k \geq l$ such that $K_{1kl} \subset \text{Int} U$;
4. for every $x \in I - A_1$ there is an open neighborhood $V$ of $x$ such that the set of pairs $(k, l)$ with $K_{1kl} \cap V \neq \emptyset$ is empty or contains only one element.

For the construction of such a family it suffices to choose in every complementary interval $(a_n, b_n)$ of the set $A_1$ two sequences of closed intervals $[c_{ink}, d_{ink}]$ and $[c_{2nk}, d_{2nk}]$ such that:

1. $c_{2n1} > d_{1n1} > c_{1n1} > d_{1n2} > c_{1n2} > \ldots \to a_n$;
2. $c_{2n1} < d_{2n1} < c_{2n2} < d_{2n2} < \ldots \to b_n$;
3. $[c_{ink}, d_{ink}] \cap A_{1+k+l} = \emptyset$, $k = 1, 2, \ldots$ and $i = 1, 2$.

(See also [8, Lemma 2.1].) For every $k = 1, 2, \ldots$ we define a continuous function $l_{ik}: K_{1k1} \to [-k, k]$ such that $l_{ik}(K_{1k1}) = [-k, k]$ and $l_{ik}(x) = 0$ if $x$ is an endpoint of the interval $K_{1k1}$.

Let

$$h_1(x) = \begin{cases} l_{ik}(x) & \text{if } x \in K_{1k1}, k = 1, 2, \ldots \\ l_1(x) & \text{in the remaining case}. \end{cases}$$

Evidently $h_1$ is a center Darboux Baire $\frac{1}{2}$ function, because it is continuous at every point $x \in I - A_1$ and $h_1/A_1$ is a Baire 1 function such that the set $h_1(A_1)$ is finite.

In the second step we choose a family $K_{2kl}$ $(k = 1, 2, \ldots$ and $l \leq k)$ of closed intervals such that:

1. $K_{2kl} \subset I - A_{2+k+l}$, $k = 1, 2, \ldots$ and $l \leq k$;
2. $K_{2kl} \cap K_{2rs} = \emptyset$ if $(k, l) \neq (r, s)$, $k, r = 1, 2, \ldots$ and $l \leq k$ and $s \leq r$;
3. $K_{2kl} \cap K_{1rs} = \emptyset$, $k, r = 1, 2, \ldots$ and $l \leq k$ and $s \leq r$;
4. for every $x \in \text{Cl}(A_2 - A_1)$ and for every closed interval $V \ni x$ and for every $l$ there is $k \geq l$ such that $K_{2kl} \subset \text{Int} V$;
5. for every $x \in I - A_2$ there is an open set $U \ni x$ such that the set of couples $(k, l)$ with $K_{2kl} \cap U \neq \emptyset$ is empty or it contains only one element.

For every $k = 2, 3, \ldots$, we define continuous functions $l_{21k}: K_{1k2} \to [-k, k]$ and $l_{22k}: K_{2k1} \to [-2^{-1}, 2^{-1}]$ such that $l_{21k}(K_{1k2}) = [-k, k]$, $l_{22k}(K_{2k1}) = [-2^{-1}, 2^{-1}]$, $l_{21k}(x) = 0$ for $x$ an endpoint of $K_{1k2}$ and $l_{22k}(x) = 0$ for $x$
an endpoint of \( K_{2k1} \). Let
\[
\begin{align*}
h_2(x) = \begin{cases} 
  l_{21k}(x) & \text{for } x \in K_{1k2}, k = 2, 3, \ldots \\
  l_{22k}(x) & \text{for } x \in K_{2k1}, k = 2, 3, \ldots \\
  l_2(x) & \text{in the remaining case.}
\end{cases}
\end{align*}
\]
The function \( h_2 \) is continuous at each point \( x \in A_2 \) and it is a Darboux Baire \( \frac{1}{2} \) function.

Generally, in step \( n > 2 \) we remark that \( K_{k1l} \cap A_n = \emptyset \) for \( i < n, k = 1, 2, \ldots, l \leq k \) with \( k + l \geq n \) and we construct a family of closed intervals \( K_{nkl}, k = 1, 2, \ldots \) and \( l \leq k \) such that:
\[
\begin{align*}
(1) & \quad K_{k1l} \subset I - A_{n+k+1}, k = 1, 2, \ldots \text{ and } l \leq k; \\
(2) & \quad K_{k1l} \cap K_{irs} = \emptyset \text{ if } i < n \text{ and } i + r + s \geq n; (k, r = 1, 2, \ldots, l \leq k, s \leq r); \\
(3) & \quad K_{k1l} \cap K_{nrk} = \emptyset \text{ if } (k, k) \neq (r, s), k, r = 1, 2, \ldots, l \leq k, s \leq r; \\
(4) & \quad \text{for every } x \in \text{Cl}(A_{n} - A_{n-1}), \text{for every closed interval } V \ni x \text{ and for every } l \text{ there exists } k \geq l \text{ such that } K_{k1l} \subset \text{Int}V; \\
(5) & \quad \text{for every } x \in I - A_{n} \text{ there is an open set } U \ni x \text{ such that the set } \{(k, l); K_{k1l} \cap U \neq \emptyset\} \text{ is empty or it contains only one element.}
\end{align*}
\]
For every pair \( (i, k) \) such that \( i + k > n \) and \( i \leq n \) we define a continuous function \( l_{nik}: K_{i,k,n-i+1} \rightarrow \mathbb{R} \) such that \( l_{nik}(x) = 0 \) for \( x \) being an endpoint of \( K_{ikn} \), \( l_{nik}(K_{ikn}) = [-k, k] \) and \( l_{nik}(K_{i,k,n-i+1}) = [-2^{-i+1}, 2^{-i+1}] \) for \( i > 1 \). Let
\[
\begin{align*}
h_{n}(x) = \begin{cases} 
  l_{nik}(x) & \text{for } x \in K_{i,k,n-i+1}, i \leq n, i + k > n \\
  l_{n}(x) & \text{in the remaining case.}
\end{cases}
\end{align*}
\]
\( h_n \) is a Darboux Baire \( \frac{1}{2} \) function, because it is continuous at every point \( x \in I - A_{n} \) and \( h_{n}/A_{n} \) is a Baire 1 function such that the set \( h_{n}(A_{n}) \) is finite and \( h_{n}(A_{k} - A_{k-1}) \subset (-2^{-k+1}, 2^{-k+1}) \) for \( k = 2, 3, \ldots, n \). Obviously
\[
\lim_{n \to \infty} h_{n}(x) = h(x) \quad \text{for } x \in \bigcup_{n=1}^{\infty} A_{n} \text{ and for } x \in I - \bigcup_{n=1}^{\infty} A_{n} - \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} K_{nkl}.
\]
Let \( x \in K_{nkl} \) for some indices \( n, k, l \). We have \( h(x) = 0 \). Let \( r > 0 \). There is an index \( m \) such that \( 2^{-m+1} < r \). For \( i = 1, \ldots, m \) let \( n(i) \) denote 1 if \( x \in I - \bigcup_{k=1}^{\infty} \bigcup_{l \leq k} K_{ikl} \) or the index \( l(x) \) such that \( x \in K_{i,k,n(x),l(x)} \) in the remaining case. Let
\[
N = 3 \max(n(1), n(2), \ldots, n(m), m).
\]
Observe that for \( n > N, x \notin K_{i,k,n-i+1} \) for \( i = 1, \ldots, m \) and \( i + k \geq n \) and
\[
|h_{n}(x)| < 2^{-m+1} < r.
\]
So \( h = \lim_{n \to \infty} h_{n} \).

There is a sequence of continuous functions \( g_{n}: I \rightarrow \mathbb{R} \) \((n = 1, 2, \ldots)\) such that \( g = \lim_{n \to \infty} g_{n} \). Since \( g_{n} \) is continuous everywhere and \( h_{n} \) is a center
Darboux Baire 1 function and \( h_n(A_n) \) is finite, so \( f_n = g_n + h_n \) is a Darboux Baire \( \frac{1}{2} \) function. We have also
\[
\lim_{n \to \infty} f_n = \lim_{n \to \infty} g_n + \lim_{n \to \infty} h_n = g + h = f
\]
and the proof is complete.

**Remark 4.** Suppose that \( f: I \to R \) is a Darboux Baire \( \frac{1}{2} \) function and \( B \subset I \) is a dense set in \( I \). From Theorem 0 it follows that for every \( x \in I \) we have
\[
\lim \inf_{t \in B} f(t) \leq f(x) \leq \lim \sup_{t \in B} f(t).
\]

From [3] we derive the following theorem:

**Theorem 2.** Suppose that \( F: I^2 \to R \) is a function such that all the sections \( F_x(t) = F(x, t) \) \((x, t \in I)\) are of Darboux Baire \( \frac{1}{2} \) class (and continuous almost everywhere). If all the sections \( F_y(t) = F(t, y) \) \((t, y \in I)\) have the Baire property (are measurable), then \( F \) has also the Baire property (is measurable).

**Example 2.** There is a nonmeasurable function \( F: I^2 \to R \) such that all the sections \( F_x, F_y \) are Darboux Baire \( \frac{1}{2} \). Let \( A \subset I \) be a Cantor set of positive measure and let \( B \subset I^2 \) be a nonmeasurable set of full exterior measure such that all the sections \( B_x = \{t \in I; (x, t) \in B\} \) and \( B_y = \{t \in I; (t, y) \in B\} \) are empty or contain one or two points [7]. In every complementary interval \( (a_n, b_n) \) of the set \( A \) we choose a closed interval \( I_n \subset (a_n, b_n) \). For every \( n = 1, 2, \ldots \) we define a continuous function \( f_n: I_n \to R \) such that \( f_n(I_n) = [0, 1] \) and \( f_n(x) = 0 \) if \( x \) is an endpoint of \( I_n \).

Define for \((x, y) \in (A \times (R - A)) \cup ((R - A) \times A)\),
\[
g(x, y) = \begin{cases} 
 f_n(y) & \text{for } x \in A \text{ and } y \in I_n, \ n = 1, 2, \ldots, \\
 f_n(x) & \text{for } y \in A \text{ and } x \in I_n, \ n = 1, 2, \ldots, \\
 0 & \text{in the remaining case.}
\end{cases}
\]

Since the set \((A \times (R - A)) \cup ((R - A) \times A)\) is closed in the space \( H = (I \times I) - (A \times A) \), so there exists a continuous function \( h: H \to [0, 1] \) such that \( h(x, y) = g(x, y) \) for \((x, y) \in (A \times (R - A)) \cup ((R - A) \times A)\).

Let
\[
F(x, y) = \begin{cases} 
 h(x, y) & \text{for } (x, y) \in H, \\
 1 & \text{for } (x, y) \in (A \times A) \cap B, \\
 0 & \text{in the remaining cases.}
\end{cases}
\]

Since the set \((A \times A) \cap B\) is not measurable, so \( F \) is nonmeasurable. Evidently all the sections \( F_x, F_y \) are in the Darboux Baire \( \frac{1}{2} \) class.

**References**


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