A MULTIVARIATE GENERALIZATION
OF A THEOREM OF R. H. FARRELL

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Abstract. Let \((\Omega, \mathcal{G}, \mu)\) denote a finite measure space and \(T_j: \Omega \rightarrow [a_j, b_j](\mathcal{B} \cap [a_j, b_j])\)-measurable functions, \(j = 1, \ldots, n\). Then the algebra of functions generated by \(1, T_1, \ldots, T_n\) is a dense subset of \(L_1(\Omega, \mathcal{G}, \mu)\) if and only if for any \(A \in \mathcal{G}\) there exists some \(B \in (T_1, \ldots, T_n)^{-1}(\mathcal{B}^n \cap [a_1, b_1] \times \cdots \times [a_n, b_n])\), such that \(\mu(A \Delta B) = 0\) is valid. In particular, this condition is satisfied if \((\Omega, \mathcal{G})\) is a Blackwell space and \((T_1, \ldots, T_n): \Omega \rightarrow [a_1, b_1] \times \cdots \times [a_n, b_n]\) is in addition one-to-one.

If for any \(A \in \mathcal{G}\) there exists some \(B \in (T_1, \ldots, T_n)^{-1}(\mathcal{B}^n \cap [a_1, b_1] \times \cdots \times [a_n, b_n])\), where \(\mathcal{B}^n\) denotes the Borel \(\sigma\)-algebra of \(\mathbb{R}^n\), such that \(\mu(A \Delta B) = 0\) holds, then for any \(f \in L_1(\Omega, \mathcal{G}, \mu)\) there exists some Borel function \(g: [a_1, b_1] \times \cdots \times [a_n, b_n] \rightarrow \mathbb{R}\), such that \(f = g \circ (T_1, \ldots, T_n)\) is valid a.e. \(\mu\). Furthermore, any \(g \in L_1([a_1, b_1] \times \cdots \times [a_n, b_n], \mathcal{B}^n \cap [a_1, b_1] \times \cdots \times [a_n, b_n], \mu^T)\), where \(T\) stands for \((T_1, \ldots, T_n)\), can be approximated by a continuous function \(h: [a_1, b_1] \times \cdots \times [a_n, b_n] \rightarrow \mathbb{R}\) with respect to the \(L_1\)-norm, and according to the approximation theorem of Bernstein any such function \(h\) can be approximated uniformly by a polynomial of \((t_1, \ldots, t_n) \in [a_1, b_1] \times \cdots \times [a_n, b_n]\), which proves that the algebra of functions generated by \(1, T_1, \ldots, T_n\) is dense in \(L_1(\Omega, \mathcal{G}, \mu)\). Conversely, if this algebra of functions is a dense subset of \(L_1(\Omega, \mathcal{G}, \mu)\), then any \(f \in L_1(\Omega, \mathcal{G}, \mu)\) can be written as \(f = g \circ (T_1, \ldots, T_n)\) a.e. \(\mu\) for some Borel function \(g: [a_1, b_1] \times \cdots \times [a_n, b_n] \rightarrow \mathbb{R}\), if one takes into consideration that \(L_1(\Omega, \mathcal{G}, \mu)\)-convergence implies \(\mu\)-stochastic convergence and any \(\mu\)-stochastic convergent sequence admits a subsequence that converges a.e. \(\mu\). In particular, for any \(A \in \mathcal{G}\) there exists some \(B \in T^{-1}(\mathcal{B}^n \cap [a_1, b_1] \times \cdots \times [a_n, b_n])\) satisfying \(\mu(A \Delta B) = 0\). Thus, the following has been proved:

Theorem. Let \((\Omega, \mathcal{G}, \mu)\) denote a finite measure space and let \(T_j: \Omega \rightarrow [a_j, b_j]\) be \((\mathcal{B}, \mathcal{B} \cap [a_j, b_j])\)-measurable, \(j = 1, \ldots, n\). Then the algebra of functions
generated by $1, T_1, \ldots, T_n$ is a dense subset of $\mathcal{L}(\Omega, \mathcal{S}, \mu)$ if and only if for any $A \in \mathcal{S}$ there exists some $B \in (T_1, \ldots, T_n)^{-1}(\mathcal{B}^n \cap [a_1, b_1] \times \cdots \times [a_n, b_n])$ satisfying $\mu(A \Delta B) = 0$.

**Example.** If $(\Omega, \mathcal{S})$ is a Blackwell space (cf. [3, p. 37]) and $(T_1, \ldots, T_n): \Omega \to [a_1, b_1] \times \cdots \times [a_n, b_n]$ is a one-to-one Borel function, then according to a theorem of Blackwell $\mathcal{S} = (T_1, \ldots, T_n)^{-1}(\mathcal{B}^n \cap [a_1, b_1] \times \cdots \times [a_n, b_n])$ holds (cf. [3, p. 38]); i.e., the algebra of functions generated by $1, T_1, \ldots, T_n$ is dense in $\mathcal{L}(\Omega, \mathcal{S}, \mu)$. For $(\Omega, \mathcal{S}) = (\mathbb{R}, \mathcal{B})$, $n = 1$, and $T_1$ strictly monotone, this theorem is due to R. H. Farrell (cf. [2] and [5, p. 518]). It is not difficult to see that in this case $T_1^{-1}(\mathcal{B} \cap [a_1, b_1]) = \mathcal{B}$ is valid. This follows immediately from $[\alpha, \beta] = T_1^{-1}([T_1(\alpha), T_1(\beta)])$, $[\alpha, \beta] \subset \mathbb{R}$ (if $T_1$ is strictly increasing), which implies $\mathcal{B} \subset T_1^{-1}(\mathcal{B} \cap [a_1, b_1])$.

**Remark.** It might be interesting to mention that the condition that for all $A \in \mathcal{S}$ there exists some $B \in \mathcal{I}$, where $\mathcal{I}$ denotes some sub-$\sigma$-algebra of $\mathcal{S}$ satisfying $\mu(A \Delta B) = 0$, is equivalent to the assertion that $\mu$ is an extremal point of the convex set $E$ consisting of all finite measures $\nu$ on $\mathcal{S}$ satisfying $\nu|\mathcal{I} = \mu|\mathcal{I}$, where $\nu|\mathcal{I}$ denotes the restriction of $\nu$ to $\mathcal{I}$. If for $\nu \in E$ the approximation property holds and $\nu = \alpha \nu_1 + (1 - \alpha) \nu_2$ for some $\nu_j \in E$, $j = 1, 2$, $0 < \alpha < 1$, is valid, then $\nu = \nu_1 = \nu_2$ follows if we choose for $A \in \mathcal{S}$ some $B \in \mathcal{I}$ with $\mu(A \Delta B) = 0$. This implies $\nu_j(A \Delta B) = 0$, $j = 1, 2$, which leads to $\nu_j(A) = \nu_j(B) = \mu(B) = \mu(A)$, $j = 1, 2$; i.e., $\nu$ is an extremal point of $E$. For a functional analytical proof of the converse direction based on a Hahn-Banach type argument cf. [1] and [4]. For sake of completeness the following short, probabilistic proof based on the notion of conditional expectation is included:

**Proof.** Without loss of generality one can assume that the finite measure $\mu$ is a probability measure $P$. Then $P_f$ defined by

$$P_f(A) = \int_A \frac{f}{E_p(f|\mathcal{I})} dP, \quad A \in \mathcal{S},$$

where $E_p(f|\mathcal{I})$ denotes the conditional expectation of $f$, $f: \Omega \to \mathbb{R}$ $\mathcal{S}$-measurable and $M \geq f \geq m > 0$ for some $m$, $M \in \mathbb{R}$, belongs to $E$. Furthermore, $P_f(A) = \int_A g dP$, $A \in \mathcal{S}$, is valid, where $g: \Omega \to \mathbb{R}$ is $\mathcal{S}$-measurable and satisfies $0 \leq g \leq L$ a.e. $P$ for some $L \geq 1$. Hence, $h = (1 - g)/L$ has the property $-1 \leq h \leq 1$ a.e. $P$ and $\int_A h dP = 0$, $A \in \mathcal{S}$. Now the assumption that some $P \in E$ is an extremal point of $E$ yields

$$P(A) = \int_A (1 + h) dP = \int_A (1 - h) dP, \quad A \in \mathcal{S},$$

from which $\int_A h dP = 0$, $A \in \mathcal{S}$; i.e., $h = 0$ a.e. $P$ follows. Therefore, $P_f = P$ holds, which leads for $f = \varepsilon + I_{A_1}$, $\varepsilon > 0$, $A_1 \in \mathcal{S}$, to

$$P(A) = \int_A \left( \frac{I_{A_1}}{E_p(I_{A_1|\mathcal{I}})} I\{E_p(I_{A_1|\mathcal{I}})>0\} + I\{E_p(I_{A_1|\mathcal{I}})=0\} \right) dP, \quad A \in \mathcal{S},$$
for $\varepsilon \to 0$. This implies $P(A_1 \Delta \{E_p(I_{A_1} | \mathcal{F}) > 0\}) = 0$; i.e., for any $A_1 \in \mathcal{F}$ there exists some $B_1 \in \mathcal{F}$, namely $B_1 = \{E_p(I_{A_1} | \mathcal{F}) > 0\}$, satisfying $P(A_1 \Delta B_1) = 0$.

REFERENCES


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