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A MULTIVARIATE GENERALIZATION OF A THEOREM OF R. H. FARRELL

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ABSTRACT. Let $(\Omega, \mathfrak{S}, \mu)$ denote a finite measure space and $T_j: \Omega \to [a_j, b_j]$ $(\mathfrak{S}, \mathfrak{B} \cap [a_j, b_j])$-measurable functions, $j = 1, \dots, n$. Then the algebra of functions generated by $1, T_1, \dots, T_n$ is a dense subset of $\mathscr{L}_1(\Omega, \mathfrak{S}, P)$ if and only if for any $A \in \mathfrak{S}$ there exists some $B \in (T_1, \dots, T_n)^{-1}(\mathfrak{B}^n \cap [a_1, b_1] \times \cdots \times [a_n, b_n])$, such that $\mu(A \triangle B) = 0$ is valid. In particular, this condition is satisfied if $(\Omega, \mathfrak{S})$ is a Blackwell space and $(T_1, \dots, T_n): \Omega \to [a_1, b_1] \times \cdots \times [a_n, b_n]$ is in addition one-to-one.

If for any $A \in \mathfrak{S}$ there exists some

$$B \in (T_1, \dots, T_n)^{-1}(\mathfrak{B}^n \cap [a_1, b_1] \times \cdots \times [a_n, b_n]),$$

where $\mathfrak{B}^n$ denotes the Borel σ-algebra of $\mathbb{R}^n$, such that $\mu(A \triangle B) = 0$ holds, then for any $f \in \mathscr{L}_1(\Omega, \mathfrak{S}, \mu)$ there exists some Borel function $g: [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}$, such that $f = g \circ (T_1, \dots, T_n)$ is valid a.e. μ. Furthermore, any $g \in \mathscr{L}_1([a_1, b_1] \times \cdots \times [a_n, b_n], \mathfrak{B}^n \cap [a_1, b_1] \times \cdots \times [a_n, b_n], \mu^T)$, where T stands for $(T_1, \dots, T_n)$, can be approximated by a continuous function $h: [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}$ with respect to the $\mathscr{L}_1$-norm, and according to the approximation theorem of Bernstein any such function h can be approximated uniformly by a polynomial of $(t_1, \dots, t_n) \in [a_1, b_1] \times \cdots \times [a_n, b_n]$, which proves that the algebra of functions generated by $1, T_1, \dots, T_n$ is dense in $\mathscr{L}_1(\Omega, \mathfrak{S}, \mu)$. Conversely, if this algebra of functions is a dense subset of $\mathscr{L}_1(\Omega, \mathfrak{S}, \mu)$, then any $f \in \mathscr{L}_1(\Omega, \mathfrak{S}, \mu)$ can be written as $f = g \circ (T_1, \dots, T_n)$ a.e. μ for some Borel function $g: [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}$, if one takes into consideration that $\mathscr{L}_1(\Omega, \mathfrak{S}, \mu)$-convergence implies μ-stochastic convergence and any μ-stochastic convergent sequence admits a subsequence that converges a.e. μ. In particular, for any $A \in \mathfrak{S}$ there exists some $B \in T^{-1}(\mathfrak{B}^n \cap [a_1, b_1] \times \cdots \times [a_n, b_n])$ satisfying $\mu(A \triangle B) = 0$. Thus, the following has been proved:

Theorem. *Let $(\Omega, \mathfrak{S}, \mu)$ denote a finite measure space and let $T_j: \Omega \to [a_j, b_j]$ be $(\mathfrak{S}, \mathfrak{B} \cap [a_j, b_j])$-measurable, $j = 1, \dots, n$. Then the algebra of functions*

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generated by $1, T_1, \dots, T_n$ *is a dense subset of* $\mathscr{L}_1(\Omega, \mathfrak{S}, \mu)$ *if and only if for any* $A \in \mathfrak{S}$ *there exists some* $B \in (T_1, \dots, T_n)^{-1}(\mathfrak{B}^n \cap [a_1, b_1] \times \cdots \times [a_n, b_n])$ *satisfying* $\mu(A \triangle B) = 0$.

Example. If $(\Omega, \mathfrak{S})$ is a Blackwell space (cf. [3, p. 37]) and $(T_1, \dots, T_n): \Omega \to [a_1, b_1] \times \cdots \times [a_n, b_n]$ is a one-to-one Borel function, then according to a theorem of Blackwell $\mathfrak{S} = (T_1, \dots, T_n)^{-1}(\mathfrak{B}^n \cap [a_1, b_1] \times \cdots \times [a_n, b_n])$ holds (cf. [3, p. 38]); i.e., the algebra of functions generated by $1, T_1, \dots, T_n$ is dense in $\mathscr{L}_1(\Omega, \mathfrak{S}, \mu)$. For $(\Omega, \mathfrak{S}) = (\mathbb{R}, \mathfrak{B})$, $n = 1$, and T_1 strictly monotone, this theorem is due to R. H. Farrell (cf. [2] and [5, p. 518]). It is not difficult to see that in this case $T_1^{-1}(\mathfrak{B} \cap [a_1, b_1]) = \mathfrak{B}$ is valid. This follows immediately from $[\alpha, \beta] = T_1^{-1}([T_1(\alpha), T_1(\beta)])$, $[\alpha, \beta] \subset \mathbb{R}$ (if T_1 is strictly increasing), which implies $\mathfrak{B} \subset T_1^{-1}(\mathfrak{B} \cap [a_1, b_1])$.

Remark. It might be interesting to mention that the condition that for all $A \in \mathfrak{S}$ there exists some $B \in \mathfrak{T}$, where $\mathfrak{T}$ denotes some sub-σ-algebra of $\mathfrak{S}$ satisfying $\mu(A \triangle B) = 0$, is equivalent to the assertion that μ is an extremal point of the convex set E consisting of all finite measures ν on $\mathfrak{S}$ satisfying $\nu|\mathfrak{T} = \mu|\mathfrak{T}$, where $\nu|\mathfrak{T}$ denotes the restriction of ν to $\mathfrak{T}$. If for $\nu \in E$ the approximation property holds and $\nu = \alpha\nu_1 + (1-\alpha)\nu_2$ for some $\nu_j \in E$, $j = 1, 2$, $0 < \alpha < 1$, is valid, then $\nu = \nu_1 = \nu_2$ follows if we choose for $A \in \mathfrak{S}$ some $B \in \mathfrak{T}$ with $\mu(A \triangle B) = 0$. This implies $\nu_j(A \triangle B) = 0$, $j = 1, 2$, which leads to $\nu_j(A) = \nu_j(B) = \mu(B) = \mu(A)$, $j = 1, 2$; i.e., ν is an extremal point of E. For a functional analytical proof of the converse direction based on a Hahn-Banach type argument cf. [1] and [4]. For sake of completeness the following short, probabilistic proof based on the notion of conditional expectation is included:

Proof. Without loss of generality one can assume that the finite measure μ is a probability measure P. Then P_f defined by

$$P_f(A) = \int_A \frac{f}{E_P(f|\mathfrak{T})} dP, \qquad A \in \mathfrak{S},$$

where $E_P(f|\mathfrak{T})$ denotes the conditional expectation of f, $f: \Omega \to \mathbb{R}$ $\mathfrak{S}$-measurable and $M \geq f \geq m > 0$ for some m, $M \in \mathbb{R}$, belongs to E. Furthermore, $P_f(A) = \int_A g\,dP$, $A \in \mathfrak{S}$, is valid, where $g: \Omega \to \mathbb{R}$ is $\mathfrak{S}$-measurable and satisfies $0 \leq g \leq L$ a.e. P for some $L \geq 1$. Hence, $h = (1-g)/L$ has the property $-1 \leq h \leq 1$ a.e. P and $\int_A h\,dP = 0$, $A \in \mathfrak{S}$. Now the assumption that some $P \in E$ is an extremal point of E yields

$$P(A) = \int_A (1+h)dP = \int_A (1-h)dP, \qquad A \in \mathfrak{S},$$

from which $\int_A h\,dP = 0$, $A \in \mathfrak{S}$; i.e., $h = 0$ a.e. P follows. Therefore, $P_f = P$ holds, which leads for $f = \varepsilon + I_{A_1}$, $\varepsilon > 0$, $A_1 \in \mathfrak{S}$, to

$$P(A) = \int_A \left(\frac{I_{A_1}}{E_P(I_{A_1}|\mathfrak{T})} I_{\{E_P(I_A|\mathfrak{T})>0\}} + I_{\{E_P(I_{A_1}|\mathfrak{T})=0\}} \right) dP, \qquad A \in \mathfrak{S},$$

for $\varepsilon \to 0$. This implies $P(A_1 \triangle \{E_P(I_{A_1}|\mathfrak{T}) > 0\}) = 0$; i.e., for any $A_1 \in \mathfrak{S}$ there exists some $B_1 \in \mathfrak{T}$, namely $B_1 = \{E_P(I_{A_1}|\mathfrak{T}) > 0\}$, satisfying $P(A_1 \triangle B_1) = 0$.

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