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NONINTEGRABILITY OF SUPERHARMONIC FUNCTIONS

NORIAKI SUZUKI

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ABSTRACT. In this article we prove the following: If u is a nonzero superharmonic function on a proper subdomain D in R^n, then

$$\int_D |u(x)|^p \delta_D(x)^{np-n-2p} dx = \infty,$$

where $0 < p \le 1$ and $\delta_D(x)$ denotes the distance between x and the boundary of D.

Let D be a domain in the euclidean space R^n $(n \ge 2)$, with $D \ne R^n$. We denote by $\delta_D(x)$ the distance between $x \in D$ and ∂D, the boundary of D. In contrast to [3], we obtain the following result:

Theorem. *Let $0 < p \le 1$. If a superharmonic function u on D in R^n satisfies*

$$(*) \qquad \int_D |u(x)|^p \delta_D(x)^{np-n-2p} dx < \infty,$$

then u vanishes identically.

It is obvious that in the condition $(*)$ the exponent $np - n - 2p$ cannot be replaced by any larger number (e.g., see the remark below). Incidentally, we note that if D is a bounded $C^{1,1}$-domain, then $\int_D s(x)^p \delta_D(x)^{np-n-p} dx = \infty$ for any nonzero subharmonic function $s \ge 0$ on D (cf. [4]).

Now, denote by $B(x, r)$ the open ball of radius $r > 0$ centered at $x \in R^n$ and by G_D the Green function of D (in the sense of [2, p. 84]) if it exists. Positive constants depending only on $\alpha, \beta, \dots$ are denoted by $c(\alpha, \beta, \dots)$ and are not necessarily the same on any two occurrences.

Lemma. *If a nonnegative subharmonic function s on D satisfies*

$$\int_D s(x)^p \delta_D(x)^{np-n-2p} dx < \infty$$

for some $0 < p \le 1$, then $s(x) = o(\delta_D(x)^{2-n})$ as $\delta_D(x) \to 0$. In particular, if $n = 2$, then s is bounded and $s(x) \to 0$ as $x \to \partial D$.

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Proof. For $x \in D$, let $B = B(x, \delta_D(x))$. By the same argument as in [1, p. 172], we obtain

$$\begin{aligned} s(x)^p &\le c(n, p)\delta_D(x)^{-n} \int_B s(y)^p \, dy \\ &\le c(n, p)\delta_D(x)^{p(2-n)} \int_B s(y)^p \delta_D(y)^{np-n-2p} \, dy, \end{aligned}$$

since $\delta_D(y) \le 2\delta_D(x)$ for $y \in B$. From this, the assertions of the lemma follow immediately.

Proof of the theorem. Define $s(x) = -\min\{u(x), 0\}$. First we assume that D has the Green function. Since $\int_D s(x)^p \delta_D(x)^{np-n-2p} dx < \infty$,

$$v_{s,p}(\cdot) := \int_D G_D(\cdot, y) s(y)^p \delta_D(y)^{np-n-2p} \, dy$$

is a potential. For any $x \in D$, set $t = \delta_D(x)/2$, $B = B(x, 2t)$, and $B_o = B(x, t)$. Then, for $y \in B_o$, $\delta_D(y) \le 3t/2$ and $s(y)^{p-1} \ge c(n, p, s) t^{(n-2)(1-p)}$ by the lemma. Hence the sub-mean-value property of s gives

$$\begin{aligned} v_{s,p}(x) &\ge \int_{B_o} G_B(x, y) s(y)^p \delta_D(y)^{np-n-2p} \, dy \\ &\ge c(n) \int_{B_o} |x - y|^{2-n} s(y)[s(y)^{p-1} \delta_D(y)^{np-n-2p}] dy \\ &\ge c(n, p, s) \int_0^t \int_{|\xi|=1} r^{2-n} s(x + r\xi) t^{-2} r^{n-1} \, dr \, d\sigma(\xi) \\ &\ge c(n, p, s) s(x). \end{aligned}$$

Since every subharmonic function dominated by a potential is nonpositive, we deduce that $s \equiv 0$; that is, $u \ge 0$ on D. On the other hand, the same argument as above leads to $v_{1,1} = \int_D G_D(\cdot, y)\delta_D(y)^{-2} \, dy \equiv \infty$, and hence any potential $v = \int_D G_D(\cdot, z) d\lambda(z)$ with $\lambda \ne 0$ satisfies

$$\begin{aligned} \int_D v(y)\delta_D(y)^{-2} \, dy &= \int_D \int_D G_D(y, z)\delta_D(y)^{-2} \, d\lambda(z) dy \\ &= \int_D \int_D G_D(z, y)\delta_D(y)^{-2} \, dy \, d\lambda(z) \\ &= \int_D v_{1,1}(z) d\lambda(z) = \infty. \end{aligned}$$

Since every positive superharmonic function has a nonzero potential as its minorant, we conclude that $u \equiv 0$ under $(*)$ by remarking that u^p is superharmonic and $-2 \ge np - n - 2p$.

Next we consider the case that D does not have the Green function. Then $n = 2$ (cf. [2, Theorem 5.12]). On account of the lemma, s is bounded and $s(x) \to 0$ as $x \to \partial D$. For $\varepsilon > 0$, set $s_\varepsilon = \max\{s - \varepsilon, 0\}$ and choose a ball $B_\varepsilon = B(x_\varepsilon, r_\varepsilon)$ in D such that $s_\varepsilon = 0$ on B_ε. For $\eta > 0$, we also set

$$s_{\varepsilon,\eta}(x) = \begin{cases} \max\{s_\varepsilon(x) - \eta \log(|x - x_\varepsilon|/r_\varepsilon), 0\}, & x \in D \backslash B_\varepsilon \\ 0, & x \in B_\varepsilon. \end{cases}$$

Then $0 \leq s_{\varepsilon,\eta} \leq s_\varepsilon \leq s$, and $s_{\varepsilon,\eta}$ is subharmonic on D. Since $s_{\varepsilon,\eta}(x) \to 0$ as $x \to \partial D$ or as x tends to infinity, the maximum principle implies that $s_{\varepsilon,\eta} \equiv 0$, so that $s(x) \leq \varepsilon + \eta|\log(|x - x_\varepsilon|/r_\varepsilon)|$. Letting $\eta \downarrow 0$ and then $\varepsilon \downarrow 0$, we deduce that $s \equiv 0$. Thus u is nonnegative, so that u is constant. The conclusion $u \equiv 0$ follows easily. This completes the proof.

Remark. Let $D = B(0, 2)\setminus\{0\}$ and $B = B(0, 1)$ and define $u(x) = -G_B(x, 0)$ on $B \cap D$ and $u(x) = 0$ on $D\setminus B$. Then u is superharmonic on D and satisfies $\int_D |u(x)|^p \delta_D(x)^{np-n-2p+\varepsilon}\, dx < \infty$ for any $\varepsilon > 0$.

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Department of Mathematics, Faculty of Science, Hiroshima University, Hiroshima 730, Japan

Current address: Department of Mathematics, College of General Education, Nagoya University, Nagoya 464, Japan