WHEN ARE TOUCHPOINTS LIMITS FOR GENERALIZED PÓLYA URNS?

ROBIN PEMANTLE

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Abstract. Hill, Lane, and Sudderth (1980) consider a Pólya-like urn scheme in which \(X_0, X_1, \ldots\), are the successive proportions of red balls in an urn to which at the \(n\)th stage a red ball is added with probability \(f(X_n)\) and a black ball is added with probability \(1 - f(X_n)\). For continuous \(f\) they show that \(X_n\) converges almost surely to a random limit \(X\) which is a fixed point for \(f\) and ask whether the point \(p\) can be a limit if \(p\) is a touchpoint, i.e. \(p = f(p)\) but \(f(x) > x\) for \(x \neq p\) in a neighborhood of \(p\). The answer is that it depends on whether the limit of \((f(x) - x)/(p - x)\) is greater or less than \(1/2\) as \(x\) approaches \(p\) from the side where \((f(x) - x)/(p - x)\) is positive.

Hill, Lane, and Sudderth (1980), hereafter referred to as [HLS], consider the following urn scheme. Let \(f:[0, 1] \rightarrow [0, 1]\) be any function and let an urn begin with \(l\) balls of which a proportion \(X_{l-1} \in (0, 1)\) are red and the remainder black. Add a new ball to the urn, whose color is red with probability \(f(X_{l-1})\) and black otherwise. Let \(X_l\) be the new proportion of red balls and iterate the procedure, producing a sequence of proportions \(X_{l-1}, X_l, X_{l+1}, \ldots\). In the case where \(f\) is continuous, they show that \(X_n\) converges almost surely to some random variable \(X\). Furthermore, \(f(X) = X\) almost surely [HLS, Theorem 2.1 and Corollary 3.1]. Categorize points \(p \in (0, 1)\) for which \(p = f(p)\) by calling them upcrossings if \((y - p)(f(y) - y)\) is positive for all \(y\) in some neighborhood of \(p\), and downcrossings if \((y - p)(f(y) - y)\) is negative for all \(y\) in some neighborhood of \(p\). The terminology comes from the way the graph \(y = f(x)\) crosses the graph \(y = x\). The next results of [HLS] are that \(\text{prob}(X_n \rightarrow p) > 0\) if \(p\) is a downcrossing and \(f\) maps \((0, 1)\) into itself, while \(\text{prob}(X_n \rightarrow p) = 0\) if \(p\) is an upcrossing. The only other kind of isolated point, \(p\), in the set \(\{x : x = f(x)\}\) is a touchpoint where \(f(y) > y\) for all \(y \neq p\) in a neighborhood of \(p\), or else \(f(y) < y\) for all \(y \neq p\) in a neighborhood of \(p\). They ask whether touchpoints can be in the support of the limiting random variable \(X\).

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This note answers their question both ways for continuous $f$, giving a condition on $f$ near $p$ implying $\text{prob}(X_n \to p) > 0$ and another condition that implies $\text{prob}(X_n \to p) = 0$. These conditions almost meet, in the sense that they cover all cases where $(f(x) - x)/(p - x)$ has a limit as $x \to p$ except for the case where the limit is equal to $1/2$. By symmetry between red and black balls, there is no loss of generality in considering only touchpoints of the first kind, where $f(y) > y$ for $y \neq p$ in a neighborhood of $p$. Therefore, the proofs will be given only for the touchpoints of the first kind. Furthermore, whether $X_n$ converges to $p$ with positive probability depends only on the germ of $f$ at $p$ [HLS, Lemma 4.1], so the arguments below will assume without loss of generality that $f(y) > y$ for all $y \neq p, 1$, as well as assuming that $f$ maps $(0, 1)$ into itself.

Let $\mathcal{F}_n$ be the $\sigma$-algebra generated by $\{X_i : i \leq n\}$, and let $\mathcal{F}_\tau$ be defined similarly for any stopping time $\tau$. The key to the proof of both conditions will be the decomposition of the submartingale $\{X_n, \mathcal{F}_n\}$ into a martingale and an increasing process. Write $X_{n+1} = X_n + A_n + Y_n$, where

$$A_n = \mathbb{E}(X_{n+1} - X_n | \mathcal{F}_n)$$

is $\mathcal{F}_n$-measurable and $Y_n = X_{n+1} - X_n - A_n$, so $\mathbb{E}(Y_n | \mathcal{F}) = 0$. Then calculate the following conditional probabilities given $\mathcal{F}_n$:

$$X_{n+1} = \begin{cases} \frac{nX_{n+1}}{n+1} = X_n + \frac{1-X_n}{n+1} & \text{with probability } f(X_n), \\ \frac{nX_n}{n+1} = X_n - \frac{X_n}{n+1} & \text{with probability } 1 - f(X_n). \end{cases}$$

This gives $A_n = (f(X_n) - X_n)/(n + 1)$, which is nonnegative by assumption; and hence

$$Y_n = \begin{cases} \frac{1-f(X_n)}{n+1} & \text{with probability } f(X_n), \\ -\frac{f(X_n)}{n+1} & \text{with probability } 1 - f(X_n). \end{cases}$$

Also, $Y_n$ is a mean zero random variable given $\mathcal{F}_n$, with the conditional distribution of $Y_n$ given $\mathcal{F}_n$ satisfying $\min(f(X_n), 1 - f(X_n))^2(n + 1)^{-2} = \inf Y_n^2 \leq \mathbb{E}(Y_n^2 | \mathcal{F}_n) \leq \sup Y_n^2 < (n + 1)^{-2}$, where the inf is over $\omega$ in the $\mathcal{F}_n$-measurable set for which $X_n$ has the given value. Defining

$$Z_{n, m} = \sum_{i=n}^{m-1} Y_i$$

yields for each fixed $n$ a martingale $\{Z_{n, m}, \mathcal{F}_m\}$ with an $L^2$-bound $\mathbb{E}Z_{n, \infty}^2 \leq \sum_{i=n}^{\infty} (i + 1)^{-2} \leq 1/n$. If $f$ is bounded away from 0 and 1 near $p$, then a lower $L^2$-bound is gotten by stopping the process $X_n$ when it exits an interval on which $\min(f(X_n), 1 - f(X_n)) > b$. If $\tau$ is any stopping time bounded above by the exit time of the interval, then the above lower bound on $\mathbb{E}Y_m^2$ gives

$$(1) \quad \mathbb{E}(Z_{n, \infty}^2 | \mathcal{F}_m) \geq \mathbb{E}(Z_{n, \tau}^2 | \mathcal{F}_n) \geq \text{prob}(\tau = \infty | \mathcal{F}_n)b^2(n + 1)^{-1}.$$
The idea will be that if \( f(x) - x \) is less than \( (p - x)/2 \), then the increasing part \( A \) pushes \( X \) toward \( p \) so slowly that by the time \( X \) gets close to \( p \), the increments of \( Z \) are very very small, and \( Z \) cannot push \( X \) above \( p \). So, in fact, one gets convergence to \( p \) from below. On the other hand, if \( f(x) - x \) is greater than \( (p - x)/2 \), then the increasing part pushes \( X \) toward \( p \) fast enough so that the increments of \( Z \) are big enough compared with \( p - X \), so that, eventually, the addition of \( Z \) puts \( X \) over \( p \). A result along the lines of Pemantle \cite{P1, P2} then implies that \( X_n \) cannot converge to \( p \).

Remark. It will be shown that convergence to a touchpoint near which \( f(x) > x \) is always from the left. Thus the behavior of the function to the right of the touchpoint is irrelevant.

**Theorem 1.** Let \( f \) be continuous in a neighborhood of a touchpoint \( p \) and suppose that \( f \) maps \((0, 1)\) into itself. Further suppose that \( x < f(x) \leq x + k(p - x) \) for some \( k < 1/2 \) and all \( x \) in some left neighborhood, \((p - \epsilon, p)\), of \( p \). Then \( \text{prob}(X_n \to p) > 0 \). [Similarly, if \( x > f(x) \geq x - k(x - p) \) for some \( k < 1/2 \) and all \( x \) in a right neighborhood, \((p, p + \epsilon)\), of \( p \), then also \( \text{prob}(X_n \to p) > 0 \).]

**Corollary 2.** If \( f \) is differentiable at a touchpoint \( p \) and continuous in a neighborhood of \( p \), then \( \text{prob}(X_n \to p) > 0 \) under the same nontriviality assumption \( f((0, 1)) \subseteq (0, 1) \).

**Proof.** Since \( f(x) - x \) does not change sign at \( p \), the derivative of \( f(x) - x \) must be zero at \( p \) and Theorem 1 applies. \( \square \)

**Proof of Theorem 1.** Replacing \( f \) by a function agreeing with \( f \) on a neighborhood of \( p \), there is no loss of generality in assuming that \( f \) is continuous and that \( f(x) > x \) for all \( x \in [0, 1) \setminus \{p\} \). Thus it will suffice to prove that with positive probability there is an \( N \) for which \( n > N \) implies \( X_n < p \), since \( X_n \) converges to a fixed point of \( f \) \cite{HLS, Corollary 3.1}, which must then be \( p \).

Pick a \( k \) for which the hypothesis is satisfied and pick \( k_1 \) with \( k < k_1 < 1/2 \). Pick a constant \( \gamma \) just barely greater than 1 so that \( \gamma k_1 < 1/2 \). The function \( g(r) = re^{(1-r)/2k_1} \) has value 1 at \( r = 1 \) and derivative \( g'(1) = 1 - 1/2k_1 \gamma < 0 \), so there is an \( r \in (0, 1) \) for which \( g(r) > 1 \). Fix such an \( r \). Define

\[
T(n) = e^{(1-r)/2k_1} \quad \text{so} \quad g(r^n) = r^nT(n)^{1/2} > 1.
\]

Choose \( M \) big enough so that \( \gamma r^M < \epsilon \) and define

\[
\tau_M = \inf\{j > T(M): X_{j-1} < p - r^M < X_j\}
\]

if such a \( j \) exists, and \( \tau_M = -\infty \) otherwise. By the nontriviality assumption that \( f \) maps \((0, 1)\) into itself, \( \text{prob}(\tau_M > T(M)) > 0 \). For each \( n \geq M \), define \( \tau_{n+1} = \inf\{j \geq \tau_n: X_{j} > p - r^{n+1}\} \). Note that if \( X_j \geq p \) for some \( j > T(M) \), then \( \tau_n \leq j \) for all \( n \geq M \). The theorem will be proved by showing that \( \text{prob}(\tau_n > T(n) \text{ for all } n \geq M) > 0 \), which will imply that with
nonzero probability, $X_n$ is eventually less than $p$, proving the theorem. Begin by assuming that $\tau_n > T(n)$ and calculate $\text{prob}(\tau_{n+1} > T(n+1) | \tau_n > T(N))$ as follows. Let $\mathcal{B}$ be the event \( \{ \inf_{j>\tau_n} X_j \geq p - \gamma r^n \} \) and estimate

$$\text{prob}(\mathcal{B}^c | \tau_n > T(N)) = \text{prob} \left( \inf_{j>\tau_n} X_j < p - \gamma r^n | \tau_n > T(N) \right)$$

$$\leq \text{prob} \left( \inf_{j>\tau_n} Z_{\tau_n, j} < -(\gamma - 1)r^n | \tau_n > T(N) \right)$$

$$\leq E(Z_{\tau_n, \infty}^2 | \tau_n > T(N))/(\gamma - 1)^2 r^{-2n}$$

$$= (\gamma - 1)^{-2} [g(r)]^{-2n}. $$

Next, note that if $\mathcal{B}$ holds, then

$$\sum_{T(n) < j < T(n+1)} A_j = \sum_{T(n) < j < T(n+1)} (f(X_j) - X_j)/(j + 1)$$

$$< (\ln[T(n+1)] - \ln[T(n)])(k\gamma r^n)$$

$$\leq (k\gamma r^n)((1 - r)/\gamma k_1 + 1/T(n))$$

$$= (k/k_1)(r^n - r^{n+1}) + k\gamma r^n/T(n).$$

But then if $\mathcal{B}$ holds and $\tau_{n+1} = L \leq T(n+1)$, it must be the case that

$$Z_{\tau_n, L} = X_L - X_{\tau_n} - \sum_{j=\tau_n}^{L-1} A_j$$

$$\geq X_L - X_{\tau_n} - \sum_{T(n) < j < T(n+1)} A_j$$

$$\geq r^n - r^{n+1} - \xi_n - (k/k_1)(r^n - r^{n+1}) - k\gamma r^n / T(n)$$

$$= r^n(1 - r)(1 - (k/k_1)) - \xi_n - k\gamma r^n / T(n)$$

$$= r^n(1 - r)(1 - (k/k_1)) - \tilde{\xi}_n.$$ 

The term $\xi_n$ comes from the fact that $X_{\tau_n}$ may overshoot the stopping point $p - r^n$, and $\tilde{\xi}_n$ denotes the sum of $\xi_n$ and the $k\gamma r^n / T(n)$ term. Then $\xi_n$ is bounded by $X_{\tau_n} - X_{\tau_n - 1} < \tau_n - T(n)^{-1}$ by assumption. Since $T(n)^{-1}$ is of order less than $r^{2n}$, the $\tilde{\xi}_n$ contribution vanishes asymptotically in the sense that

$$\frac{r^n(1 - r)(1 - (k/k_1)) - \tilde{\xi}_n}{r^n(1 - r)(1 - (k/k_1))} \to 1.$$
Now $E(Z_{\tau_n,\infty}^2 \mid \tau_n > T(N)) < T(n)^{-1}$, so

$$\text{prob}(\tau_{n+1} \leq T(n+1) \mid \tau_n > T(N))$$

$$\leq \text{prob}(\mathcal{B}^c \mid \tau_n > T(N))$$

$$+ \text{prob} \left( \mathcal{B} \text{ and } \sup L Z_{\tau_n,L} \geq r^n (1-r) \left( 1 - \frac{k}{k_1} \right) - \varepsilon_n \mid \tau_n > T(N) \right)$$

$$\leq (1-\gamma)^{-2} [g(r)]^{-2n} + T(n)^{-1} \left[ r^n (1-r) \left( 1 - \frac{k}{k_1} \right) - \varepsilon_n \right]^2$$

$$\leq (1-\gamma)^{-2} [g(r)]^{-2n} + [(1-r)(1-(k/k_1))]^{-2} [g(r)]^{-2n}$$

$$\times \left[ r^n (1-r) \left( 1 - \frac{k}{k_1} \right) - \varepsilon_n \right]^2.$$
Proof. Pick $b > 0$ and $N$ a neighborhood of $p$ such that $f(N) \subseteq [b, 1-b]$. Assume that $X_n \in N$ or else the result is trivially true. For $k > 0$, let $\tau \leq \infty$ be the first time $X_j$ exits $N$ or $Z_{n,j}$ exits $(-kn^{-1/2}, kn^{-1/2})$. Then equation (1) gives $E(Z_{n,j}^2 | \mathcal{F}_n) \geq \text{prob}(\tau = \infty | \mathcal{F}_n) b^2 (n+1)^{-1}$. On the other hand, $E(Z_{n,j}^2 | \mathcal{F}_n) \leq E(X_{\tau} - X_n)^2 \leq k^2/n$, since $Z$ is just the martingale part of $X$. Putting these together gives $\text{prob}(\tau = \infty | \mathcal{F}_n) \leq k^2(n+1)/b^2 n$, and choosing $k$ small enough makes this at most $1/3$. Let

$$q = \text{prob}(\tau < \infty, X_\tau \not\in N | \mathcal{F}_n),$$

so that the conditional probability of $Z_{n,j}$ exiting $(-kn^{-1/2}, kn^{-1/2})$ given $\mathcal{F}_n$ is at least $2/3 - q$. Any martingale $\mathcal{M}$ started at zero that exits an interval $(-L, L)$ with probability at least $r$ and has increments bounded by $L/2$ satisfies $\text{prob}(\sup \mathcal{M} \geq L/2) \geq (3r - 1)/4$; stopping $\mathcal{M}$ upon exiting $(-L, L/2)$ and letting $s = \text{prob}(\sup \mathcal{M} > L/2)$ gives $0 = E\mathcal{M} \leq sL + (r-s)(-L) + (1-r)(L/2) = 2L(s - (3r - 1)/4)$. Thus $Z_{n,j} \geq k/2\sqrt{n}$ for some $j$ with probability at least $(1 - 3q)/4$.

Now for any $j$, condition on the event $Z_{n,j} \geq k/2\sqrt{n}$; then the conditional probability of the event $Z_{n,\infty} < k/4\sqrt{n}$ can be bounded away from 1 using the following one-sided Tschebysheff estimate:

**Lemma 5.** If $\mathcal{M}$ has mean zero and $L < 0$, then

$$\text{prob}(\mathcal{M} \leq L) \leq \frac{E\mathcal{M}^2}{(E\mathcal{M}^2 + L^2)}.$$

**Proof.** Write $w$ for $\text{prob}(\mathcal{M} < L)$. From

$$0 = E\mathcal{M}^2 = wE(\mathcal{M} | \mathcal{M} \leq L) + (1-w)E(\mathcal{M} | \mathcal{M} > L)$$

and $E(\mathcal{M} | M \leq L) \leq L$, it is immediate that

$$E(\mathcal{M} | \mathcal{M} > L) \geq -L \frac{w}{1-w}.$$  

Then

$$E\mathcal{M}^2 = wE(M^2 | M \leq L) + (1-w)E(M^2 | \mathcal{M} > L) \geq wL^2 + (1-w)(E\mathcal{M} | \mathcal{M} > L)^2 \geq wL^2 + (1-w)E(\mathcal{M}^2 | \mathcal{M} > L)^2 \geq L^2 w/(1-w),$$

from which the desired conclusion follows.  

Apply this to the process $Z_{j,i}$ stopped at the entrance time $\tau$ of the interval $(-\infty, -k/4\sqrt{n})$ to get

$$\text{prob}(Z_{n,\infty} \leq k/4\sqrt{n} | \mathcal{F}_j) \leq \text{prob}(Z_{j,\tau} \leq -k/4\sqrt{n} | \mathcal{F}_j) \leq EZ_{j,\tau}^2/(EZ_{j,\tau}^2 + k^2/16n) \leq EZ_{n,\infty}^2/(EZ_{n,\infty}^2 + k^2/16n) \leq 16/(k^2 + 16).$$
Combining this with the previous result shows that the conditional probability of $Z_{n,\infty} > k/4\sqrt{n}$ given $\mathcal{F}_n$ is at least $(1 - 3q)k^2/(64 + 4k^2)$. Recall that $q$ is the conditional probability of the process exiting $\mathcal{N}$ given $\mathcal{F}_n$, so that the probability we are trying to bound below is at least the maximum of $q$ and $(1 - 3q)k^2/(64 + 4k^2)$. For any value of $q$ the maximum is at least $k^2/(64 + 7k^2)$, thus the statement of the lemma is proved with $c = k/4$ and $a = k^2/(64 + 7k^2)$. □

Let $\tau$ be any finite stopping time. Conditioning on $\mathcal{F}_\tau$ then gives a stopping time version of the previous lemma:

(2) $\text{prob}(Z_{\tau,\infty} > c\tau^{-1/2} \text{ or } X_{\tau+j} \notin \mathcal{N} \text{ for some } j|\mathcal{F}_\tau) > a$.

A corollary of this is a sort of converse to the proof of Theorem 1, saying that if $X_n \to p$ then it does so from the left.

**Corollary 6.** Let $p$ be a touchpoint of the first kind, i.e. $f(y) > y$ for all $y \neq p$ in a neighborhood of $p$. Then the probability of the event that either $X_n > p$ finitely often or $X_n$ does not converge to $p$ is 1.

**Proof.** Suppose to the contrary that the probability that $X_n$ converges to $p$ and is greater than or equal to $p$ infinitely often is nonzero. Then there are $n, M$, and some event $\mathcal{B} \in \mathcal{F}_n$ such that $n < M$ and conditional on $\mathcal{B}$, the probability of $X_j$ converging to $p$ and being greater than $p$ some time before $M$ but never leaving $\mathcal{N}$ after time $n$ is at least $1 - a/3$. Define $\tau$ to be the minimum of $M$ and the least $j > n$ such that $X_j > p$. Then letting $\mathcal{C}$ be the event that $X_j$ converges to $p$ without leaving $\mathcal{N}$ after time $n$,

$$\text{prob}(\tau < M|\mathcal{B})\text{prob}(\mathcal{C}|\mathcal{B}, \tau < M) + \text{prob}(\tau = M|\mathcal{B})\text{prob}(\mathcal{C}|\mathcal{B}, \tau = M) = \text{prob}(\mathcal{C}|\mathcal{B}) \geq 1 - a/3.$$ 

So

$$\text{prob}(\mathcal{C}|\mathcal{B}, \tau < M) \geq 1 - a/3 - \text{prob}(\tau = M|\mathcal{B}) \geq 1 - 2a/3.$$ 

Now $\tau < M$ implies that $X_j > p$. But since $A_n$ is an increasing process, it follows that $X_j \to p$ and $X_{\tau} > p$ together imply $Z_{\tau,\infty} < 0$. Thus

$$\text{prob}(Z_{\tau,\infty} < 0 \text{ and } X_{n+j} \in \mathcal{N} \text{ for all } j|\mathcal{B}, \tau < M) \geq 1 - 2a/3,$$

and hence

$$\text{prob}(Z_{\tau,\infty} > c\tau^{-1/2} \text{ or } X_{\tau+j} \notin \mathcal{N} \text{ for some } j|\mathcal{B}, \tau > M) \leq 2a/3.$$ 

But this contradicts (2), since the events $\mathcal{B}$ and $\tau < M$ are both in $\mathcal{F}_\tau$. □

**Continuation of the proof of Theorem 3.** It remains to show that under the hypothesis of the theorem, the probability is zero that $X_n$ eventually resides in $(p - \delta, p)$. If the probability were nonzero, then for any $\delta$ there would be an
event $\mathcal{B}$ in some $\mathcal{F}_M$ for which $\Pr(X_{M+j} \in (p - \epsilon, p))$ for all $j \geq 0, \mathcal{B} > 1 - \delta$. In fact, conditioning on $X_M$, $\mathcal{B}$ may be taken to determine $X_M$. So it suffices to show that the probability of the event $X_{M+j} \in (p - \epsilon, p)$ for all $j \geq 0$ given $X_M$ is bounded away from 1. For what follows condition on $\mathcal{F}_M$ and on $X_M \in (p - \epsilon, p)$. Also choose $M$ large enough so that for any $n > M$, $n^{-k/2k_1} < cn^{-1/2}$ where $c$ is chosen as in Lemma 4, and choose $\epsilon$ small enough so that $(p - \epsilon, p)$ is a subset of a neighborhood $\mathcal{N}$ to which Lemma 4 applies.

Begin by setting up constants and stopping times: pick a $k < 3/4$ for which the hypothesis of the theorem is satisfied and pick $k_1$ so that $k > k_1 > 1/2$. For $n \geq M$ define

$$V_n = \left(\frac{k}{k_1}\right) \ln(n) + 2 \ln(p - X_n) \quad \text{for } X_n < p \quad \text{and } -\infty \text{ otherwise.}$$

By assumption on $X_M$, $V_M > -\infty$. Let $\tau$ be the least $n \geq M$ such that $X_n \not\in (p - \epsilon, p)$ or $V_n < 0$. Observe that if $V_n > 0$ then $1/n < (p - X_n)^{2k_1/k} \leq (p - X_n)^{4/3}$, so $|X_{n+1} - X_n|$ is small compared to $p - X_n$, so $V_{\tau \wedge n}$ can never reach $-\infty$ and is in fact bounded below by $\min(-1, V_M)$. Now for $n < \tau$ calculate

$$E(\ln(p - X_{n+1}) | \mathcal{F}_n) \leq \ln E(p - X_{n+1} | \mathcal{F}_n)$$

$$= \ln(p - X_{n+1} - A_{n+1})$$

$$\leq \ln((p - X_n)(1 - k/(n + 1)))$$

$$= \ln(p - X_n) + \ln(1 - k/(n + 1));$$

so

$$E(V_{n+1} | \mathcal{F}_n) \leq V_n + \left(\frac{k}{k_1}\right) \ln(n + 1) - \ln(n)) + 2 \ln(1 - k/(n + 1))$$

$$= V_n + \left(\frac{k}{k_1}\right) (n^{-1} + o(n^{-1})) - 2k(n^{-1} + o(n^{-1}))$$

$$= V_n - (2 - 1/k_1)k + o(1)n = V_n - Cn$$

for large $n$ and some $C > 0$. So $V_{\tau \wedge \t}$ is a supermartingale for large $n$, bounded below by $\min(-1, V_M)$, and hence converges almost surely. Clearly it cannot converge without stopping, since the increments of the expectation sum to $-\infty$, therefore the stopping time is reached almost surely.

In other words, conditional upon any event in any $\mathcal{F}_M$, the probability is 1 that for some $n > M$, either $X_n$ will leave $(p - \epsilon, p)$ or $(k/k_1) \ln(n) < -2 \ln(p - X_n)$. Let $\sigma \leq \infty$ be the least $n > M$ for which $(k/k_1) \ln(n) < -2 \ln(p - X_n)$. We have just shown that the conditional probability of some $X_n$ leaving $(p - \epsilon, p)$ given $\sigma = \infty$ is one. On the other hand, the conditional probability of some $X_{n+j}$ leaving $(p - \epsilon, p)$ given $\sigma = n < \infty$ is at least $a$ by Lemma 4 since $X_{n+j} \not\in \mathcal{N}$ trivially implies $X_{n+j} \not\in (p - \epsilon, p)$, while $Z_{n, \infty} > cn^{1/2}$ implies $Z_{n, n+j} > cn^{1/2} > n^{-k/2k_1} > p - X_n$ for some $j$, which implies $X_{n+j} > p$. □
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Department of Statistics, University of California, Berkeley, California 94720

Current address: Department of Mathematics, Kidder Hall, Oregon State University, Corvallis, Oregon 97331-4005