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WHEN ARE TOUCHPOINTS LIMITS FOR GENERALIZED PÓLYA URNS?

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ABSTRACT. Hill, Lane, and Sudderth (1980) consider a Pólya-like urn scheme in which $X_0, X_1, \ldots,$ are the successive proportions of red balls in an urn to which at the n th stage a red ball is added with probability $f(X_n)$ and a black ball is added with probability $1 - f(X_n)$. For continuous f they show that X_n converges almost surely to a random limit X which is a fixed point for f and ask whether the point p can be a limit if p is a touchpoint, i.e. $p = f(p)$ but $f(x) > x$ for $x \neq p$ in a neighborhood of p. The answer is that it depends on whether the limit of $(f(x) - x)/(p - x)$ is greater or less than $1/2$ as x approaches p from the side where $(f(x) - x)/(p - x)$ is positive.

Hill, Lane, and Sudderth (1980), hereafter referred to as [HLS], consider the following urn scheme. Let $f: [0, 1] \to [0, 1]$ be any function and let an urn begin with l balls of which a proportion $X_{l-1} \in (0, 1)$ are red and the remainder black. Add a new ball to the urn, whose color is red with probability $f(X_{l-1})$ and black otherwise. Let X_l be the new proportion of red balls and iterate the procedure, producing a sequence of proportions $X_{l-1}, X_l, X_{l+1}, \ldots$. In the case where f is continuous, they show that X_n converges almost surely to some random variable X. Furthermore, $f(X) = X$ almost surely [HLS, Theorem 2.1 and Corollary 3.1]. Categorize points $p \in (0, 1)$ for which $p = f(p)$ by calling them *upcrossings* if $(y - p)(f(y) - y)$ is positive for all y in some neighborhood of p, and *downcrossings* if $(y - p)(f(y) - y)$ is negative for all y in some neighborhood of p. The terminology comes from the way the graph $y = f(x)$ crosses the graph $y = x$. The next results of [HLS] are that $\text{prob}(X_n \to p) > 0$ if p is a downcrossing and f maps $(0, 1)$ into itself, while $\text{prob}(X_n \to p) = 0$ if p is an upcrossing. The only other kind of isolated point, p, in the set $\{x: x = f(x)\}$ is a *touchpoint* where $f(y) > y$ for all $y \neq p$ in a neighborhood of p, or else $f(y) < y$ for all $y \neq p$ in a neighborhood of p. They ask whether touchpoints can be in the support of the limiting random variable X.

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This note answers their question both ways for continuous f, giving a condition on f near p implying $\operatorname{prob}(X_n \to p) > 0$ and another condition that implies $\operatorname{prob}(X_n \to p) = 0$. These conditions almost meet, in the sense that they cover all cases where $(f(x)-x)/(p-x)$ has a limit as $x \uparrow p$ except for the case where the limit is equal to 1/2. By symmetry between red and black balls, there is no loss of generality in considering only touchpoints of the first kind, where $f(y) > y$ for $y \neq p$ in a neighborhood of p. Therefore, the proofs will be given only for the touchpoints of the first kind. Furthermore, whether X_n converges to p with positive probability depends only on the germ of f at p [HLS, Lemma 4.1], so the arguments below will assume without loss of generality that $f(y) > y$ for all $y \neq p, 1$, as well as assuming that f maps $(0, 1)$ into itself.

Let $\mathscr{F}_n$ be the σ-algebra generated by $\{X_i : i \leq n\}$, and let $\mathscr{F}_\tau$ be defined similarly for any stopping time τ. The key to the proof of both conditions will be the decomposition of the submartingale $\{X_n, \mathscr{F}_n\}$ into a martingale and an increasing process. Write $X_{n+1} = X_n + A_n + Y_n$, where

$$A_n = \mathbf{E}(X_{n+1} - X_n | \mathscr{F}_n)$$

is $\mathscr{F}_n$-measurable and $Y_n = X_{n+1} - X_n - A_n$, so $\mathbf{E}(Y_n | \mathscr{F}_n) = 0$. Then calculate the following conditional probabilities given $\mathscr{F}_n$:

$$X_{n+1} = \begin{cases} \frac{nX_n+1}{n+1} = X_n + \frac{1-X_n}{n+1} & \text{with probability } f(X_n), \\ \frac{nX_n}{n+1} = X_n - \frac{X_n}{n+1} & \text{with probability } 1 - f(X_n). \end{cases}$$

This gives $A_n = (f(X_n) - X_n)/(n+1)$, which is nonnegative by assumption; and hence

$$Y_n = \begin{cases} \frac{1-f(X_n)}{n+1} & \text{with probability } f(X_n), \\ -\frac{f(X_n)}{n+1} & \text{with probability } 1 - f(X_n). \end{cases}$$

Also, Y_n is a mean zero random variable given $\mathscr{F}_n$, with the conditional distribution of Y_n given $\mathscr{F}_n$ satisfying $\min(f(X_n), 1 - f(X_n))^2 (n+1)^{-2} = \inf Y_n^2 \leq \mathbf{E}(Y_n^2 | \mathscr{F}_n) \leq \sup_\omega Y_n^2 \leq (n+1)^{-2}$, where the inf is over ω in the $\mathscr{F}_n$-measurable set for which X_n has the given value. Defining

$$Z_{n,m} = \sum_{i=n}^{m-1} Y_i$$

yields for each fixed n a martingale $\{Z_{n,m}, \mathscr{F}_m\}$ with an L^2-bound $\mathbf{E}Z_{n,\infty}^2 \leq \sum_{i=n}^\infty (i+1)^{-2} \leq 1/n$. If f is bounded away from 0 and 1 near p, then a lower L^2-bound is gotten by stopping the process X_n when it exits an interval on which $\min(f(X_n), 1-f(X_n)) > b$. If τ is any stopping time bounded above by the exit time of the interval, then the above lower bound on $\mathbf{E}Y_m^2$ gives

$$\mathbf{E}(Z_{n,\infty}^2 | \mathscr{F}_n) \geq \mathbf{E}(Z_{n,\tau}^2 | \mathscr{F}_n) \geq \operatorname{prob}(\tau = \infty | \mathscr{F}_n) b^2 (n+1)^{-1}. \tag{1}$$

The idea will be that if $f(x) - x$ is less than $(p - x)/2$, then the increasing part A pushes X toward p so slowly that by the time X gets close to p, the increments of Z are very very small, and Z cannot push X above p. So, in fact, one gets convergence to p from below. On the other hand, if $f(x) - x$ is greater than $(p - x)/2$, then the increasing part pushes X toward p fast enough so that the increments of Z are big enough compared with $p - X$, so that, eventually, the addition of Z puts X over p. A result along the lines of Pemantle [P1, P2] then implies that X_n cannot converge to p.

Remark. It will be shown that convergence to a touchpoint near which $f(x) > x$ is always from the left. Thus the behavior of the function to the right of the touchpoint is irrelevant.

Theorem 1. *Let f be continuous in a neighborhood of a touchpoint p and suppose that f maps $(0, 1)$ into itself. Further suppose that $x < f(x) \leq x + k(p - x)$ for some $k < 1/2$ and all x in some left neighborhood, $(p - \varepsilon, p)$, of p. Then* $\text{prob}(X_n \to p) > 0$. *[Similarly, if $x > f(x) \geq x - k(x - p)$ for some $k < 1/2$ and all x in a right neighborhood, $(p, p + \varepsilon)$ of p, then also* $\text{prob}(X_n \to p) > 0$.]

Corollary 2. *If f is differentiable at a touchpoint p and continuous in a neighborhood of p, then* $\text{prob}(X_n \to p) > 0$ *under the same nontriviality assumption* $f((0, 1)) \subseteq (0, 1)$.

Proof. Since $f(x) - x$ does not change sign at p, the derivative of $f(x) - x$ must be zero at p and Theorem 1 applies. □

Proof of Theorem 1. Replacing f by a function agreeing with f on a neighborhood of p, there is no loss of generality in assuming that f is continuous and that $f(x) > x$ for all $x \in [0, 1)\backslash\{p\}$. Thus it will suffice to prove that with positive probability there is an N for which $n > N$ implies $X_n < p$, since X_n converges to a fixed point of f [HLS, Corollary 3.1], which must then be p. Pick a k for which the hypothesis is satisfied and pick k_1 with $k < k_1 < 1/2$. Pick a constant γ just barely greater than 1 so that $\gamma k_1 < 1/2$. The function $g(r) = re^{(1-r)/2k_1\gamma}$ has value 1 at $r = 1$ and derivative $g'(1) = 1 - 1/2k_1\gamma < 0$, so there is an $r \in (0, 1)$ for which $g(r) > 1$. Fix such an r. Define

$$T(n) = e^{n(1-r)/\gamma k_1}, \qquad \text{so } g(r)^n = r^n T(n)^{1/2} > 1.$$

Choose M big enough so that $\gamma r^M < \varepsilon$ and define

$$\tau_M = \inf\{j > T(M) : X_{j-1} < p - r^M < X_j\}$$

if such a j exists, and $\tau_M = -\infty$ otherwise. By the nontriviality assumption that f maps $(0, 1)$ into itself, $\text{prob}(\tau_M > T(M)) > 0$. For each $n \geq M$, define $\tau_{n+1} = \inf\{j \geq \tau_n : X_j > p - r^{n+1}\}$. Note that if $X_j \geq p$ for some $j > T(M)$, then $\tau_n \leq j$ for all $n \geq M$. The theorem will be proved by showing that $\text{prob}(\tau_n > T(n)$ for all $n \geq M) > 0$, which will imply that with nonzero probability, X_n is eventually less than p, proving the theorem. Begin by assuming that $\tau_n > T(n)$ and calculate $\text{prob}(\tau_{n+1} > T(n+1) | \tau_n > T(N))$ as follows. Let $\mathscr{B}$ be the event $\{\inf_{j>\tau_n} X_j \geq p - \gamma r^n\}$ and estimate

$$\begin{aligned}
\text{prob}(\mathscr{B}^c | \tau_n > T(N)) &= \text{prob}\left(\inf_{j>\tau_n} X_j < p - \gamma r^n | \tau_n > T(N)\right) \\
&\leq \text{prob}\left(\inf_{j>\tau_n} Z_{\tau_n, j} < -(\gamma - 1) r^n | \tau_n > T(N)\right) \\
&\leq \mathbf{E}(Z^2_{\tau_n, \infty} | \tau_n > T(N))/((\gamma - 1) r^n)^2 \\
&\leq e^{-n(1-r)/k_1\gamma} (\gamma - 1)^{-2} r^{-2n} \\
&= (\gamma - 1)^{-2} [g(r)]^{-2n}.
\end{aligned}$$

Next, note that if $\mathscr{B}$ holds, then

$$\begin{aligned}
\sum_{T(n)<j<T(n+1)} A_j &= \sum_{T(n)<j<T(n+1)} (f(X_j) - X_j)/(j+1) \\
&< (\ln\lceil T(n+1)\rceil - \ln\lceil T(n)\rceil)(k\gamma r^n) \\
&\leq (k\gamma r^n)[(1-r)/\gamma k_1 + 1/T(n)] \\
&= (k/k_1)(r^n - r^{n+1}) + k\gamma r^n / T(n).
\end{aligned}$$

But then if $\mathscr{B}$ holds and $\tau_{n+1} = L \leq T(n+1)$, it must be the case that

$$\begin{aligned}
Z_{\tau_n, L} &= X_L - X_{\tau_n} - \sum_{j=\tau_n}^{L-1} A_j \\
&\geq X_L - X_{\tau_n} - \sum_{T(n)<j<T(n+1)} A_j \\
&\geq r^n - r^{n+1} - \xi_n - (k/k_1)(r^n - r^{n+1}) - k\gamma r^n / T(n) \\
&= r^n (1-r)(1 - (k/k_1)) - \xi_n - k\gamma r^n / T(n) \\
&= r^n (1-r)(1 - (k/k_1)) - \tilde{\xi}_n.
\end{aligned}$$

The term ξ_n comes from the fact that X_{τ_n} may overshoot the stopping point $p - r^n$, and $\tilde{\xi}_n$ denotes the sum of ξ_n and the $k\gamma r^n / T(n)$ term. Then ξ_n is bounded by $X_{\tau_n} - X_{\tau_n - 1} < \tau_n^{-1} < T(n)^{-1}$ by assumption. Since $T(n)^{-1}$ is of order less than r^{2n}, the $\tilde{\xi}_n$ contribution vanishes asymptotically in the sense that

$$\frac{r^n (1-r)(1 - (k/k_1)) - \tilde{\xi}_n}{r^n (1-r)(1 - (k/k_1))} \to 1.$$

Now $\mathbf{E}(Z^2_{\tau_n,\infty}|\tau_n > T(N)) < T(n)^{-1}$, so

$$\begin{aligned}
&\operatorname{prob}(\tau_{n+1} \le T(n+1)|\tau_n > T(N)) \\
&\quad \le \operatorname{prob}(\mathscr{B}^c|\tau_n > T(N)) \\
&\quad\quad + \operatorname{prob}\left(\mathscr{B} \text{ and } \sup_L Z_{\tau_n,L} \ge r^n(1-r)\left(1-\frac{k}{k_1}\right) - \tilde{\xi}_n|\tau_n > T(N)\right) \\
&\quad \le (1-\gamma)^{-2}[g(r)]^{-2n} + T(n)^{-1} \Big/ \left[r^n(1-r)\left(1-\frac{k}{k_1}\right) - \tilde{\xi}_n\right]^2 \\
&\quad \le (1-\gamma)^{-2}[g(r)]^{-2n} + [(1-r)(1-(k/k_1))]^{-2}[g(r)]^{-2n} \\
&\quad\quad \times \left[\frac{r^n(1-r)(1-(k/k_1)) - \tilde{\xi}_n}{r^n(1-r)\left(1-\frac{k}{k_1}\right)}\right]^2 .
\end{aligned}$$

Because the last term of the numerator vanishes asymptotically, the sum of these probabilities converges. Then $\operatorname{prob}(\tau_n > T(n)$ for all $n > M) = \operatorname{prob}(\tau_M > T(M)) \prod_{n \ge M}(1 - \operatorname{prob}(\tau_{n+1} \le T(n+1)|\tau_n > T(N))) > 0$ since each factor is positive and $\sum \operatorname{prob}(\tau_{n+1} \le T(n)|\tau_n > T(N))$ is finite. In this case, X_n must converge to p from below. □

Theorem 3. *Suppose that $f(x) \ge x + k(p-x)$ for some $k > 1/2$ and all x in some left neighborhood, $(p-\varepsilon, p)$, of p. Then* $\operatorname{prob}(X_n \to p) = 0$. [*Similarly, if $f(x) \le x - k(x-p)$ for some $k > 1/2$ and all x in a right neighborhood, $(p, p+\varepsilon)$ of p, then also* $\operatorname{prob}(X_n \to p) = 0$.]

Remark. No continuity assumptions are needed this time.

Proof. Again there is no loss of generality in assuming that $f(x) \ge x$ for all x; similarly, assume $f(x) \ge \min(1, x + k|p-x|)$ on $[0, p]$. Furthermore, Lemma 2.2 of [HLS] says that replacing f by a pointwise smaller function gives a process which can be defined on the same probability space so as always to be smaller. Thus replacing f by the minimum of 1 and $x + k|p-x|$ on $[0, p]$ and by x on $[p, 1]$ gives a process which converges to p whenever the original process does, so it suffices to prove the theorem for this choice of f. The importance of assuming this lies only in getting f bounded away from 0 and 1 near p (without assuming continuity) so that there will be a lower L^2-bound on Z.

The following argument is self-contained, but the reader may wish to look at Pemantle [P2, Lemmas 1 and 2] to see the template from which this proof was constructed.

Lemma 4. *There are constants $a, c > 0$ and a neighborhood $\mathscr{N}$ of p such that for any n*

$$\operatorname{prob}(Z_{n,\infty} > cn^{-1/2} \text{ or } X_{n+j} \notin \mathscr{N} \text{ for some } j|\mathscr{F}_n) > a.$$

Proof. Pick $b > 0$ and $\mathcal{N}$ a neighborhood of p such that $f(\mathcal{N}) \subseteq [b, 1-b]$. Assume that $X_n \in \mathcal{N}$ or else the result is trivially true. For $k > 0$, let $\tau \leq \infty$ be the first time X_j exits $\mathcal{N}$ or $Z_{n,j}$ exits $(-kn^{-1/2}, kn^{-1/2})$. Then equation (1) gives $\mathbf{E}(Z^2_{n,\tau}|\mathcal{F}_n) \geq \text{prob}(\tau = \infty|\mathcal{F}_n)b^2(n+1)^{-1}$. On the other hand, $\mathbf{E}(Z^2_{n,\tau}|\mathcal{F}_n) \leq \mathbf{E}(X_\tau - X_n)^2 \leq k^2/n$, since Z is just the martingale part of X. Putting these together gives $\text{prob}(\tau = \infty|\mathcal{F}_n) \leq k^2(n+1)/b^2 n$, and choosing k small enough makes this at most 1/3. Let

$$q = \text{prob}(\tau < \infty,\ X_\tau \notin \mathcal{N}|\mathcal{F}_n),$$

so that the conditional probability of $Z_{n,j}$ exiting $(-kn^{-1/2}, kn^{-1/2})$ given $\mathcal{F}_n$ is at least $2/3 - q$. Any martingale $\mathcal{M}$ started at zero that exits an interval $(-L, L)$ with probability at least r and has increments bounded by $L/2$ satisfies $\text{prob}(\sup\mathcal{M} \geq L/2) \geq (3r-1)/4$; stopping $\mathcal{M}$ upon exiting $(-L, L/2)$ and letting $s = \text{prob}(\sup\mathcal{M} > L/2)$ gives $0 = \mathbf{E}\mathcal{M} \leq sL + (r-s)(-L) + (1-r)(L/2) = 2L(s - (3r-1)/4)$. Thus $Z_{n,j} \geq k/2\sqrt{n}$ for some j with probability at least $(1-3q)/4$.

Now for any j, condition on the event $Z_{n,j} \geq k/2\sqrt{n}$; then the conditional probability of the event $Z_{n,\infty} < k/4\sqrt{n}$ can be bounded away from 1 using the following one-sided Tschebysheff estimate:

Lemma 5. *If* $\mathcal{M}$ *has mean zero and* $L < 0$, *then*

$$\text{prob}(\mathcal{M} \leq L) \leq \mathbf{E}\mathcal{M}^2/(\mathbf{E}\mathcal{M}^2 + L^2).$$

Proof. Write w for $\text{prob}(\mathcal{M} \leq L)$. From

$$0 = \mathbf{E}\mathcal{M}^2 = w\mathbf{E}(\mathcal{M}|\mathcal{M} \leq L) + (1-w)\mathbf{E}(\mathcal{M}|\mathcal{M} > L)$$

and $\mathbf{E}(\mathcal{M}|M \leq L) \leq L$, it is immediate that

$$\mathbf{E}(\mathcal{M}|\mathcal{M} > L) \geq -L\frac{w}{1-w}.$$

Then

$$\begin{aligned}\mathbf{E}\mathcal{M}^2 &= w\mathbf{E}(M^2|M \leq L) + (1-w)\mathbf{E}(\mathcal{M}^2|\mathcal{M} > L)\\ &\geq wL^2 + (1-w)(\mathbf{E}(\mathcal{M}|\mathcal{M} > L))^2\\ &\geq wL^2 + (1-w)L^2(w^2/(1-w)^2)\\ &= L^2 w/(1-w),\end{aligned}$$

from which the desired conclusion follows. □

Apply this to the process $Z_{j,i}$ stopped at the entrance time τ of the interval $(-\infty, -k/4\sqrt{n})$ to get

$$\begin{aligned}\text{prob}(Z_{n,\infty} \leq k/4\sqrt{n}|\mathcal{F}_j) &\leq \text{prob}(Z_{j,\tau} \leq -k/4\sqrt{n}|\mathcal{F}_j)\\ &\leq \mathbf{E}Z^2_{j,\tau}/(\mathbf{E}Z^2_{j,\tau} + k^2/16n)\\ &\leq \mathbf{E}Z^2_{n,\infty}/(\mathbf{E}Z^2_{n,\infty} + k^2/16n)\\ &\leq 16/(k^2+16).\end{aligned}$$

Combining this with the previous result shows that the conditional probability of $Z_{n,\infty} > k/4\sqrt{n}$ given $\mathscr{F}_n$ is at least $(1-3q)k^2/64+4k^2)$. Recall that q is the conditional probability of the process exiting $\mathscr{N}$ given $\mathscr{F}_n$, so that the probability we are trying to bound below is at least the maximum of q and $(1-3q)k^2/(64+4k^2)$. For any value of q the maximum is at least $k^2/(64+7k^2)$, thus the statement of the lemma is proved with $c = k/4$ and $a = k^2/(64+7k^2)$. □

Let τ be any finite stopping time. Conditioning on $\mathscr{F}_\tau$ then gives a stopping time version of the previous lemma:

$$\text{prob}(Z_{\tau,\infty} > c\tau^{-1/2} \text{ or } X_{\tau+j} \notin \mathscr{N} \text{ for some } j|\mathscr{F}_\tau) > a. \tag{2}$$

A corollary of this is a sort of converse to the proof of Theorem 1, saying that if $X_n \to p$ then it does so from the left.

Corollary 6. *Let p be a touchpoint of the first kind, i.e. $f(y) > y$ for all $y \neq p$ in a neighborhood of p. Then the probability of the event that either $X_n > p$ finitely often or X_n does not converge to p is 1.*

Proof. Suppose to the contrary that the probability that X_n converges to p and is greater than or equal to p infinitely often is nonzero. Then there are n, M, and some event $\mathscr{B} \in \mathscr{F}_n$ such that $n < M$ and conditional on $\mathscr{B}$, the probability of X_j converging to p and being greater than p some time before M but never leaving $\mathscr{N}$ after time n is at least $1 - a/3$. Define τ to be the minimum of M and the least $j \geq n$ such that $X_j > p$. Then letting $\mathscr{C}$ be the event that X_j converges to p without leaving $\mathscr{N}$ after time n,

$$\begin{aligned}\text{prob}(\tau < M|\mathscr{B})\text{prob}(\mathscr{C}|\mathscr{B}, \tau < M) + \text{prob}(\tau = M|\mathscr{B})\text{prob}(\mathscr{C}|\mathscr{B}, \tau = M)\\ = \text{prob}(\mathscr{C}|\mathscr{B}) \geq 1 - a/3.\end{aligned}$$

So

$$\text{prob}(\mathscr{C}|\mathscr{B}, \tau < M) \geq 1 - a/3 - \text{prob}(\tau = M|\mathscr{B}) \geq 1 - 2a/3.$$

Now $\tau < M$ implies that $X_\tau > p$. But since A_n is an increasing process, it follows that $X_j \to p$ and $X_\tau > p$ together imply $Z_{\tau,\infty} < 0$. Thus

$$\text{prob}(Z_{\tau,\infty} < 0 \text{ and } X_{n+j} \in \mathscr{N} \text{ for all } j|\mathscr{B}, \tau < M) \geq 1 - 2a/3,$$

and hence

$$\text{prob}(Z_{\tau,\infty} > c\tau^{-1/2} \text{ or } X_{\tau+j} \notin \mathscr{N} \text{ for some } j|\mathscr{B}, \tau > M) \leq 2a/3.$$

But this contradicts (2), since the events $\mathscr{B}$ and $\tau < M$ are both in $\mathscr{F}_\tau$. □

Continuation of the proof of Theorem 3. It remains to show that under the hypothesis of the theorem, the probability is zero that X_n eventually resides in $(p - \varepsilon, p)$. If the probability were nonzero, then for any δ there would be an event $\mathscr{B}$ in some $\mathscr{F}_M$ for which $\text{prob}(X_{M+j} \in (p-\varepsilon, p)$ for all $j \geq 0 | \mathscr{B}) > 1-\delta$. In fact, conditioning on X_M, $\mathscr{B}$ may be taken to determine X_M. So it suffices to show that the probability of the event $X_{M+j} \in (p-\varepsilon, p)$ for all $j \geq 0$ given X_M is bounded away from 1. For what follows condition on $\mathscr{F}_M$ and on $X_M \in (p-\varepsilon, p)$. Also choose M large enough so that for any $n > M$, $n^{-k/2k_1} < cn^{-1/2}$ where c is chosen as in Lemma 4, and choose ε small enough so that $(p-\varepsilon, p)$ is a subset of a neighborhood $\mathscr{N}$ to which Lemma 4 applies.

Begin by setting up constants and stopping times: pick a $k < 3/4$ for which the hypothesis of the theorem is satisfied and pick k_1 so that $k > k_1 > 1/2$. For $n \geq M$ define

$$V_n = (k/k_1)\ln(n) + 2\ln(p - X_n) \quad \text{for } X_n < p \quad \text{and } -\infty \text{ otherwise.}$$

By assumption on X_M, $V_M > -\infty$. Let τ be the least $n \geq M$ such that $X_n \notin (p-\varepsilon, p)$ or $V_n < 0$. Observe that if $V_n > 0$ then $1/n < (p-X_n)^{2k_1/k} \leq (p-X_n)^{4/3}$, so $|X_{n+1} - X_n|$ is small compared to $p - X_n$, so $V_{\tau\wedge n}$ can never reach $-\infty$ and is in fact bounded below by $\min(-1, V_M)$. Now for $n < \tau$ calculate

$$\begin{aligned}
\mathbf{E}(\ln(p - X_{n+1})|\mathscr{F}_n) &\leq \ln \mathbf{E}(p - X_{n+1}|\mathscr{F}_n) \\
&= \ln(p - X_n - A_n) \\
&\leq \ln((p - X_n)(1 - k/(n+1))) \\
&= \ln(p - X_n) + \ln(1 - k/(n+1));
\end{aligned}$$

so

$$\begin{aligned}
\mathbf{E}(V_{n+1}|\mathscr{F}_n) &\leq V_n + (k/k_1)(\ln(n+1) - \ln(n)) + 2\ln(1 - k/(n+1)) \\
&= V_n + (k/k_1)(n^{-1} + o(n^{-1})) - 2k(n^{-1} + o(n^{-1})) \\
&= V_n - ((2 - 1/k_1)k + o(1))n^{-1} < V_n - Cn^{-1}
\end{aligned}$$

for large n and some $C > 0$. So $V_{n\wedge\tau}$ is a supermartingale for large n, bounded below by $\min(-1, V_M)$, and hence converges almost surely. Clearly it cannot converge without stopping, since the increments of the expectation sum to $-\infty$, therefore the stopping time is reached almost surely.

In other words, conditional upon any event in any $\mathscr{F}_M$, the probability is 1 that for some $n > M$, either X_n will leave $(p-\varepsilon, p)$ or $(k/k_1)\ln(n) < -2\ln(p - X_n)$. Let $\sigma \leq \infty$ be the least $n > M$ for which $(k/k_1)\ln(n) < -2\ln(p - X_n)$. We have just shown that the conditional probability of some X_n leaving $(p-\varepsilon, p)$ given $\sigma = \infty$ is one. On the other hand, the conditional probability of some X_{n+j} leaving $(p-\varepsilon, p)$ given $\sigma = n < \infty$ is at least a by Lemma 4 since $X_{n+j} \notin \mathscr{N}$ trivially implies $X_{n+j} \notin (p-\varepsilon, p)$, while $Z_{n,\infty} > cn^{1/2}$ implies $Z_{n,n+j} > cn^{1/2} > n^{-k/2k_1} > p - X_n$ for some j, which implies $X_{n+j} > p$. □

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