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DISTRIBUTIVE LATTICES HAVING n-PERMUTABLE CONGRUENCES

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ABSTRACT. Distributive lattices having n-permutable congruences are characterized by the property that they have no n-element chain in their poset of prime ideals.

1. INTRODUCTION

During the course of investigations into various congruence properties of distributive lattices, p-algebras, double p-algebras, and de Morgan algebras in [1, 2, 5], we are led to consider, for integers n with $1 \leq n \leq 4$, the class $\mathscr{D}_n$ of all distributive lattices having no $(n+1)$-element chain in their poset of prime ideals. In [2], the members of $\mathscr{D}_n$, for any $n \geq 1$, are characterized by a sentence in the first-order theory of distributive lattices (see Proposition 3.1). $\mathscr{D}_n$ contains the lattice reducts of the members of several important varieties of algebras arising in the study of many-valued logics; including the variety of Łukasiewicz algebras of order $n+1$ and the variety $\mathscr{L}_{n+1}$ of L-algebras generated by the $(n+1)$-element chain algebra (see [3, 4]). Earlier research shows that those distributive lattices, p-algebras, and double p-algebras having permutable congruences are precisely the ones whose lattice reduct belongs to $\mathscr{D}_1$, $\mathscr{D}_2$, and $\mathscr{D}_3$, respectively (see [2, 5]). Furthermore, those distributive lattices, p-algebras, double p-algebras, and de Morgan algebras whose compact congruences are all principal are precisely the ones whose lattice reduct belongs to $\mathscr{D}_2$, $\mathscr{D}_3$, $\mathscr{D}_4$, and $\mathscr{D}_4$, respectively (see [1, 2, 5]). It is natural to ask if the members of $\mathscr{D}_n$ can be characterized by some property of their congruences. In this note we answer this question affirmatively by showing that $\mathscr{D}_n$ is the class of all distributive lattices having $(n+1)$-permutable congruences.

2. PRELIMINARIES

Henceforth, n will denote an arbitrary integer ≥ 1. If θ, ψ are binary relations on a set L then $\theta \circ \psi$ will denote their relational product and $\theta \circ^n \psi$ will stand for the compound relational product $\theta \circ \psi \circ \theta \circ \cdots$, involving n factors, starting with θ and thereafter alternating between θ and ψ. In the event that

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$n \geq 2$ and $\theta \circ^n \psi = \psi \circ^n \theta$ we will say that θ and ψ n-permute (or are *n-permutable*). If θ and ψ 2-permute we will simply say that they *permute* (or are *permutable*). An algebra is said to have *n-permutable congruences* if every pair of congruences on it n-permute.

If L is a lattice $a, b, c, d \in L$, $b \leq a$ and $d \leq c$ then we will indicate that the quotient a/b is perspective to the quotient c/d by writing $a/b \sim c/d$ and denote by $\theta(a, b)$ the principal congruence on L collapsing a/b.

For all other notation and terminology, we refer the reader to [6].

3. The class $\mathscr{D}_n$

The following was proved in [2]:

Proposition 3.1. *A distributive lattice L belongs to $\mathscr{D}_n$ iff, given any $x_i \in L$ with $0 \leq i \leq n+1$ and $x_0 \leq x_1 \leq \cdots \leq x_{n+1}$, there exist $x_i' \in L$ for $1 \leq i \leq n$ such that*

$$x_0 = x_1 \wedge x_1', \quad x_i \vee x_i' = x_{i+1} \wedge x_{i+1}'$$

when $1 \leq i < n$ and $x_n \vee x_n' = x_{n+1}$.

In order to pave the way for our new characterization of the members of $\mathscr{D}_n$ we first prove the following:

Lemma 3.2. *Congruences θ and ψ of a lattice $(n+1)$-permute iff the binary relations $\theta \cap \leq$ and $\psi \cap \leq$ $(n+1)$-permute.*

Proof. Throughout, let

$$\alpha = \begin{cases} \theta, & \text{if } n \text{ is even} \\ \psi, & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad \alpha' \in \{\theta, \psi\} \backslash \{\alpha\}.$$

Suppose that θ and ψ $(n+1)$-permute and let $(x, y) \in (\theta \cap \leq) \circ^{n+1} (\psi \cap \leq)$. Then there exist $x_i \in L$, $1 \leq i \leq n$, such that $x \leq x_1 \leq \cdots \leq x_n \leq y$ and $x \equiv x_1(\theta)$, $x_1 \equiv x_2(\psi), \ldots, x_n \equiv y(\alpha)$. Therefore there exist $x_i' \in L$, $1 \leq i \leq n$, such that

$$x \equiv x_1'(\psi), x_1' \equiv x_2'(\theta), \ldots, x_n' \equiv y(\alpha').$$

For $1 \leq i \leq n$, define $\overline{x}_i \in [x, y]$ by $\overline{x}_i = x \vee (x_1' \wedge y)$. Clearly, $x \equiv \overline{x}_1(\psi)$, $\overline{x}_1 \equiv \overline{x}_2(\theta), \ldots, \overline{x}_n \equiv y(\alpha')$. Now define elements $X_i' \in L$, for $1 \leq i \leq n$, by $X_1' = \overline{x}_1$ and $X_{k+1}' = X_k' \vee \overline{x}_{k+1}$ whenever $1 \leq k < n$. Clearly

$$x \leq X_1' \leq X_2' \leq \cdots \leq X_n' \leq y$$

and $x \equiv X_1'(\psi)$, $X_1' \equiv X_2'(\theta), \ldots, X_n' \equiv y(\alpha')$. Therefore $(x, y) \in (\psi \cap \leq) \circ^{n+1} (\theta \cap \leq)$. Similarly, $(\psi \cap \leq) \circ^{n+1} (\theta \cap \leq) \subseteq (\theta \cap \leq) \circ^{n+1} (\psi \cap \leq)$.

Suppose now that $\theta \cap \leq$ and $\psi \cap \leq$ $(n+1)$-permute. Let $(x, y) \in \theta \circ^{n+1} \psi$. Then there exist $x_i \in L$, $1 \leq i \leq n$, such that $x \equiv x_1(\theta)$, $x_1 \equiv x_2(\psi), \ldots, x_n \equiv y(\alpha)$. Define elements $X_i \in L$, $1 \leq i \leq n$, by $X_1 = x \vee x_1$ and $X_{k+1} = X_k \vee x_{k+1}$, whenever $1 \leq k < n$, and let $Y = X_n \vee y$. Clearly,

$$x \leq X_1 \leq X_2 \leq \cdots \leq X_n \leq Y$$

and $x \equiv X_1(\theta)$, $X_1 \equiv X_2(\psi), \ldots, X_n \equiv Y(\alpha)$; in other words, $(x, Y) \in (\theta \cap \leq) \circ^{n+1} (\psi \cap \leq)$. Therefore $(x, Y) \in (\psi \cap \leq) \circ^{n+1} (\theta \cap \leq)$ and so there exist $x_i' \in L$, $1 \leq i \leq n$, such that

$$x \leq x_1' \leq x_2' \leq \cdots \leq x_n' \leq Y$$

and $x \equiv x_1'(\psi)$, $x_1' \equiv x_2'(\theta), \ldots, x_n' \equiv Y(\alpha')$. Now define elements $Y_i \in L$, $1 \leq i \leq n$, by $Y_1 = x_n \vee y$ and $Y_{k+1} = x_{n-k} \vee Y_k$ whenever $1 \leq k < n$. Observe that $Y = x \vee Y_n$,

$$y \leq Y_1 \leq Y_2 \leq \cdots \leq Y_n \leq Y$$

and $y \equiv Y_1(\alpha), \ldots, Y_{n-1} \equiv Y_n(\psi)$, $Y_n \equiv Y(\theta)$. Therefore there exist $y_i' \in L$, $1 \leq i \leq n$, such that

$$y \leq y_1' \leq \cdots \leq y_{n-1}' \leq y_n' \leq Y$$

and

$$y \equiv y_1'(\alpha'), \ldots, y_{n-1}' \equiv y_n'(\theta),\ y_n' \equiv Y(\psi).$$

Now,

$$\begin{aligned} x = x \wedge Y &\equiv x_1' \wedge y_n'(\psi), \\ x_1' \wedge y_n' &\equiv x_2' \wedge y_{n-1}'(\theta), \\ x_2' \wedge y_{n-1}' &\equiv x_3' \wedge y_{n-2}'(\psi), \\ &\vdots \\ x_{n-1}' \wedge y_2' &\equiv x_n' \wedge y_1'(\alpha), \quad \text{and} \\ x_n' \wedge y_1' &\equiv Y \wedge y(\alpha') = y. \end{aligned}$$

Thus $(x, y) \in \psi \circ^{n+1} \theta$. Similarly, we can show that $\psi \circ^{n+1} \theta \subseteq \theta \circ^{n+1} \psi$ and so θ and ψ $(n+1)$-permute. □

A well-known consequence of Proposition 3.1 is that a distributive lattice belongs to $\mathscr{D}_1$ iff it is relatively complemented. It is also known that a distributive lattice has permutable congruences iff it is relatively complemented. This yields the case where $n = 1$ in the following:

Theorem 3.3. *Let L be a distributive lattice. Then $L \in \mathscr{D}_n$ iff it has $(n+1)$-permutable congruences.*

Proof. Suppose that $n \geq 2$ and $L \in \mathscr{D}_n$. Let θ, ψ be congruences of L and $(x, y) \in (\theta \cap \leq) \circ^{n+1} (\psi \cap \leq)$. For the sake of notational convenience, let us suppose that n is odd. Then there exist $x_i \in L$ for $1 \leq i \leq n$ such that

$$x = x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} = y$$

and

$$x_0 \equiv x_1(\theta),\ x_1 \equiv x_2(\psi), \ldots, x_n \equiv x_{n+1}(\psi).$$

By Proposition 3.1, there are $x_i' \in L$ for $1 \leq i \leq n$ such that $x = x_0 = x_1 \wedge x_1'$, $x_i \vee x_i' = x_{i+1} \wedge x_{i+1}'$, for $1 \leq i \leq n$, and $x_n \vee x_n' = x_{n+1} = y$. Let us write

$z_0 = x$ and $z_i = x_i \vee x_i'$ when $1 \le i \le n$. Observe that $z_1/x_1 \sim x_1'/x$ and $x_1 \equiv z_1(\psi)$, so that $x \equiv x_1'(\psi)$, and furthermore, $x_n/z_{n-1} \sim y/x_n'$ and $z_{n-1} \equiv x_n(\theta)$, so that $x_n' \equiv y(\theta)$. Suppose now that i is odd and $3 \le i \le n$. Then $x_{i-1}/z_{i-2} \sim z_{i-1}/x_{i-1}'$ and $z_{i-2} \equiv x_{i-1}(\psi)$ so that $x_{i-1}' \equiv z_{i-1}(\psi)$, and furthermore $z_i/x_i \sim x_i'/z_{i-1}$ and $x_i \equiv z_i(\psi)$ so that $z_{i-1} \equiv x_i'(\psi)$. Therefore $x_{i-1}' \equiv x_i'(\psi)$. It follows that

$$x \equiv x_1'(\psi),\, x_2' \equiv x_3'(\psi),\, \ldots,\, x_{n-1}' \equiv x_n'(\psi).$$

In the event that i is even and $2 \le i \le n-1$, a similar argument yields $x_{i-1}' \equiv x_i'(\theta)$. Consequently,

$$x_1' \equiv x_2'(\theta),\, x_3' \equiv x_4'(\theta),\, \ldots,\, x_n' \equiv y(\theta).$$

Thus, $(x, y) \in (\psi \cap \le) \circ^{n+1} (\theta \cap \le)$ under the assumption that n is odd. The case where n is even is dealt with similarly and so, for any $n \ge 2$ and arbitrary congruences θ, ψ on L, $(\theta \cap \le) \circ^{n+1} (\psi \cap \le) \subseteq (\psi \cap \le) \circ^{n+1} (\theta \cap \le)$. Therefore, by Lemma 3.2, L has $(n+1)$-permutable congruences.

Suppose now that L has $(n+1)$-permutable congruences. For $0 \le i \le n+1$, let $x_i \in L$ satisfy $x_0 \le x_1 \le \cdots \le x_n \le x_{n+1}$. Initially, let us suppose that n is odd and so $n = 2k+1$, for some $k \ge 1$. Define congruences θ and ψ of L by

$$\theta = \bigvee_{i=0}^{k} \theta(x_{2i}, x_{2i+1}) \quad \text{and} \quad \psi = \bigvee_{j=0}^{k} \theta(x_{2j+1}, x_{2j+2}).$$

First, observe that $\theta \wedge \psi = \bigvee_{i,j=0}^{k} \theta(x_{2i}, x_{2i+1}) \wedge \theta(x_{2j+1}, x_{2j+2})$, since the congruence lattice of L is distributive, that $x_{2i} \le x_{2i+1} \le x_{2j+1} \le x_{2j+2}$ when $i \le j$, and that $x_{2j+1} \le x_{2j+2} \le x_{2i} \le x_{2i+1}$ when $i > j$. However, it is easy to see, using the well-known description of principal congruences of distributive lattices, that if $a, b, c, d \in L$ and $a \le b \le c \le d$ then $\theta(a, b) \wedge \theta(c, d) = \omega$. Therefore $\theta \wedge \psi = \omega$. Now, $x_0 \equiv x_1(\theta)$, $x_1 \equiv x_2(\psi), \ldots, x_n \equiv x_{n+1}(\psi)$ and so there exist $x_i' \in L$, for $1 \le i \le n$, such that

$$x_0 \le x_1' \le \cdots \le x_n' \le x_{n+1}$$

and $x_0 \equiv x_1'(\psi)$, $x_1' \equiv x_2'(\theta), \ldots, x_n' \equiv x_{n+1}(\theta)$, by Lemma 3.2. Let us define, for $1 \le i \le n$, $X_i' = (x_{i-1} \vee x_i') \wedge x_{i+1}$ and note that $X_i' \in [x_{i-1}, x_{i+1}]$.

We claim that $x_1 \wedge X_1' = x_0$. Indeed, $x_1 \wedge X_1' = x_1 \wedge [(x_0 \vee x_1') \wedge x_2] = (x_0 \vee x_1') \wedge x_1 = x_1' \wedge x_1$ so that $x_1' \wedge X_1' \equiv x_1' \wedge x_0(\theta) = x_0$ and $x_1 \wedge X_1' \equiv x_0 \wedge x_1(\psi) = x_0$. Therefore $x_1 \wedge X_1' \equiv x_0(\theta \wedge \psi)$ and so $x_1 \wedge X_1' = x_0$.

Next we show that $x_n \vee X_n' = x_{n+1}$. Indeed, $x_n \vee X_n' = x_n \vee [(x_{n-1} \vee x_n') \wedge x_{n+1}] = x_n \vee x_n'$ so $x_n \vee X_n' \equiv x_{n+1} \vee x_n'(\psi) = x_{n+1}$ and $x_n \vee X_n' \equiv x_n \vee x_{n+1}(\theta) = x_{n+1}$. Therefore $x_n \vee X_n' \equiv x_{n+1}(\theta \wedge \psi)$ and so $x_n \vee X_n' = x_{n+1}$.

It remains to show that $x_i \vee X_i' = x_{i+1} \wedge X_{i+1}'$, for $1 \le i < n$. Observe that, for $1 \le i < n$, $x_i \vee X_i' = x_i \vee [(x_{i-1} \vee x_i') \wedge x_{i+1}] = (x_i \vee x_i') \wedge x_{i+1}$

and suppose that i is odd. Then $x_i \equiv x'_{i+1}(\psi)$ and $x_i \equiv x'_{i+1}(\theta)$. Therefore, $x_i \vee X'_i \equiv (x_i \vee x'_{i+1}) \wedge x_{i+1}(\theta) = x_{i+1} \wedge [(x_i \vee x'_{i+1}) \wedge x_{i+2}] = x_{i+1} \wedge X'_{i+1}$; that is, $x_i \vee X'_i \equiv x_{i+1} \wedge X'_{i+1}(\theta)$. Furthermore, $x_i \vee X'_i \equiv (x_i \vee x'_i) \wedge x_i(\psi) = x_i$ and $x_{i+1} \wedge X'_{i+1} = x_{i+1} \wedge [(x_i \vee x_{i+1}) \wedge x_{i+2}] = x_{i+1} \wedge (x'_i \vee x_{i+1}) \equiv x_i \wedge (x_i \vee x'_{i+1})(\psi) = x_i$ so $x_i \vee X'_i \equiv x_{i+1} \wedge X'_{i+1}(\psi)$. Therefore, $x_i \vee X'_i \equiv x_{i+1} \wedge X'_{i+1}(\theta \wedge \psi)$ and so $x_i \vee X'_i = x_{i+1} \wedge X'_{i+1}$. When i is even and $1 \leq i < n$ we have $x_i \equiv x_{i+1}(\theta)$ and $x'_i \equiv x'_{i+1}(\psi)$ and an argument similar to the one where i is odd also yields $x_i \vee X'_i \equiv x_{i+1} \wedge X'_{i+1}(\theta \wedge \psi)$. Thus, $x_i \vee X'_i = x_{i+1} \wedge X'_{i+1}$, for any i with $1 \leq i < n$.

In the event that n is even, so that $n = 2k$ for some $k \geq 1$, we define

$$\theta = \bigvee_{i=0}^{k} \theta(x_{2i}, x_{2i+1}) \quad \text{and} \quad \psi = \bigvee_{j=1}^{k} \theta(x_{2j-1}, x_{2j}).$$

Again we can show that $\theta \wedge \psi = \omega$, proceeding in a manner similar to that for which n is odd and draw the same conclusions. In any case, $L \in \mathscr{D}_n$ by Proposition 3.1. □

Corollary 3.4. *The length of the poset of prime ideals of a distributive lattice is n iff L has $(n+2)$-permutable but not $(n+1)$-permutable congruences.*

Corollary 3.5. *Every compact congruence of a distributive lattice is principal iff it has 3-permutable congruences.*

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