DISTRIBUTIVE LATTICES HAVING \( n \)-PERMUTABLE CONGRUENCES

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Abstract. Distributive lattices having \( n \)-permutable congruences are characterized by the property that they have no \( n \)-element chain in their poset of prime ideals.

1. Introduction

During the course of investigations into various congruence properties of distributive lattices, \( p \)-algebras, double \( p \)-algebras, and de Morgan algebras in [1, 2, 5], we are led to consider, for integers \( n \) with \( 1 \leq n \leq 4 \), the class \( \mathcal{D}_n \) of all distributive lattices having no \( (n+1) \)-element chain in their poset of prime ideals. In [2], the members of \( \mathcal{D}_n \), for any \( n \geq 1 \), are characterized by a sentence in the first-order theory of distributive lattices (see Proposition 3.1). \( \mathcal{D}_n \) contains the lattice reducts of the members of several important varieties of algebras arising in the study of many-valued logics; including the variety of Lukasiewicz algebras of order \( n + 1 \) and the variety \( \mathcal{L}_{n+1} \) of \( L \)-algebras generated by the \( (n+1) \)-element chain algebra (see [3, 4]). Earlier research shows that those distributive lattices, \( p \)-algebras, and double \( p \)-algebras having permutable congruences are precisely the ones whose lattice reduct belongs to \( \mathcal{D}_1, \mathcal{D}_2, \) and \( \mathcal{D}_3 \), respectively (see [2, 5]). Furthermore, those distributive lattices, \( p \)-algebras, double \( p \)-algebras, and de Morgan algebras whose compact congruences are all principal are precisely the ones whose lattice reduct belongs to \( \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \) and \( \mathcal{D}_4 \), respectively (see [1, 2, 5]). It is natural to ask if the members of \( \mathcal{D}_n \) can be characterized by some property of their congruences. In this note we answer this question affirmatively by showing that \( \mathcal{D}_n \) is the class of all distributive lattices having \( (n+1) \)-permutable congruences.

2. Preliminaries

Henceforth, \( n \) will denote an arbitrary integer \( \geq 1 \). If \( \theta, \psi \) are binary relations on a set \( L \) then \( \theta \circ \psi \) will denote their relational product and \( \theta \circ^n \psi \) will stand for the compound relational product \( \theta \circ \psi \circ \theta \circ \cdots \), involving \( n \) factors, starting with \( \theta \) and thereafter alternating between \( \theta \) and \( \psi \). In the event that
n \geq 2 \text{ and } \theta \circ^n \psi = \psi \circ^n \theta \text{ we will say that } \theta \text{ and } \psi \text{ n-permute (or are n-permutable). If } \theta \text{ and } \psi \text{ 2-permute we will simply say that they permute (or are permutable). An algebra is said to have n-permutable congruences if every pair of congruences on it n-permute.}

If \( L \) is a lattice \( a, b, c, d \in L \), \( b \leq a \) and \( d \leq c \) then we will indicate that the quotient \( a/b \) is perspective to the quotient \( c/d \) by writing \( a/b \sim c/d \) and denote by \( \theta(a, b) \) the principal congruence on \( L \) collapsing \( a/b \).

For all other notation and terminology, we refer the reader to [6].

3. The class \( \mathbb{D}_n \)

The following was proved in [2]:

**Proposition 3.1.** A distributive lattice \( L \) belongs to \( \mathbb{D}_n \) iff, given any \( x_i \in L \) with \( 0 \leq i \leq n+1 \) and \( x_0 \leq x_1 \leq \cdots \leq x_{n+1} \), there exist \( x'_i \in L \) for \( 1 \leq i \leq n \) such that

\[
x_0 = x_1 \land x'_1, \quad x_i \lor x'_i = x_{i+1} \land x'_{i+1}
\]

when \( 1 \leq i < n \) and \( x_n \lor x'_n = x_{n+1} \).

In order to pave the way for our new characterization of the members of \( \mathbb{D}_n \) we first prove the following:

**Lemma 3.2.** Congruences \( \theta \) and \( \psi \) of a lattice \((n+1)\)-permute iff the binary relations \( \theta \cap \leq \) and \( \psi \cap \leq \) \((n+1)\)-permute.

**Proof.** Throughout, let

\[
\alpha = \begin{cases} 
\theta, & \text{if } n \text{ is even} \\
\psi, & \text{if } n \text{ is odd} 
\end{cases} \quad \text{and} \quad \alpha' = \{ \theta, \psi \} \setminus \{ \alpha \}.
\]

Suppose that \( \theta \) and \( \psi \) \((n+1)\)-permute and let \((x, y) \in (\theta \cap \leq) \circ^{n+1} (\psi \cap \leq) \). Then there exist \( x_i \in L \), \( 1 \leq i \leq n \), such that \( x \leq x_1 \leq \cdots \leq x_n \leq y \) and \( x \equiv x_1(\theta) \), \( x_1 \equiv x_2(\psi) \), \( \ldots \), \( x_n \equiv y(\alpha) \). Therefore there exist \( x_i' \in L \), \( 1 \leq i \leq n \), such that

\[
x \equiv x'_1(\psi), \quad x'_1 \equiv x'_2(\theta), \quad \ldots \quad x'_n \equiv y(\alpha').
\]

For \( 1 \leq i \leq n \), define \( \bar{x}_i \in [x, y] \) by \( \bar{x}_i = x \lor (x'_i \land y) \). Clearly, \( x \equiv \bar{x}_1(\psi) \), \( \bar{x}_1 \equiv \bar{x}_2(\theta) \), \( \ldots \), \( \bar{x}_n \equiv y(\alpha') \). Now define elements \( X'_i \in L \), for \( 1 \leq i \leq n \), by \( X'_1 = \bar{x}_1 \) and \( X'_{k+1} = X'_k \lor \bar{x}_{k+1} \) whenever \( 1 \leq k < n \). Clearly

\[
x \leq X'_1 \leq X'_2 \leq \cdots \leq X'_n \leq y
\]

and \( x \equiv X'_1(\psi), \quad X'_1 \equiv X'_2(\theta), \quad \ldots \quad X'_n \equiv y(\alpha') \). Therefore \((x, y) \in (\psi \cap \leq) \circ^{n+1} (\theta \cap \leq) \). Similarly, \((y \cap \leq) \circ^{n+1} (\theta \cap \leq) \subseteq (\theta \cap \leq) \circ^{n+1} (\psi \cap \leq) \).

Suppose now that \( \theta \cap \leq \) and \( \psi \cap \leq \) \((n+1)\)-permute. Let \((x, y) \in \theta \circ^{n+1} \psi \). Then there exist \( x_i \in L \), \( 1 \leq i \leq n \), such that \( x \equiv x_1(\theta) \), \( x_1 \equiv x_2(\psi) \), \( \ldots \), \( x_n \equiv y(\alpha) \). Define elements \( X'_i \in L \), \( 1 \leq i \leq n \), by \( X'_1 = x \lor x_1 \) and \( X'_{k+1} = X'_k \lor x_{k+1} \) whenever \( 1 \leq k < n \), and let \( Y = X'_n \lor y \). Clearly,

\[
x \leq X_1 \leq X_2 \leq \cdots \leq X_n \leq Y
\]
and \( x \equiv X_1(\theta), \ldots, X_n \equiv Y(\alpha) \); in other words, \((x, Y) \in (\theta \cap \leq) \circ^{n+1} (\psi \cap \leq)\). Therefore \((x, Y) \in (\psi \cap \leq) \circ^{n+1} (\theta \cap \leq)\) and so there exist \(x_i' \in L, 1 \leq i \leq n\), such that

\[
x \leq x_1' \leq x_2' \leq \cdots \leq x_n' \leq Y
\]

and \( x \equiv x_1'(\psi), \ldots, x_n' \equiv Y(\alpha) \). Now define elements \( Y_1 \in L, 1 \leq i \leq n\), by \( Y_1 = x_n \cup y \) and \( Y_{k+1} = x_{n-k} \cup Y_k \) whenever \( 1 \leq k < n\). Observe that \( Y = x \cup Y_n \),

\[
y \leq Y_1 \leq Y_2 \leq \cdots \leq Y_n \leq Y
\]

and \( y \equiv Y_1(\alpha), \ldots, Y_{n-1} \equiv Y_n(\psi), Y \equiv Y(\theta) \). Therefore there exist \( y_i' \in L, 1 \leq i \leq n\), such that

\[
y \leq y_1' \leq \cdots \leq y_{n-1}' \leq y_n' \leq Y
\]

and

\[
y \equiv y_1'(\alpha), \ldots, y_{n-1}' \equiv y_n'(\theta), y_n' \equiv Y(\psi).
\]

Now,

\[
x = x \wedge Y \equiv x_1' \wedge y_n'(\psi),
\]

\[
x_1' \wedge y_n' \equiv x_2' \wedge y_{n-1}'(\theta),
\]

\[
x_2' \wedge y_{n-1}' \equiv x_3' \wedge y_{n-2}'(\psi),
\]

\[
\vdots
\]

\[
x_{n-1}' \wedge y_2' \equiv x_n' \wedge y_1'(\alpha), \quad \text{and}
\]

\[
x_n' \wedge y_1' \equiv Y \wedge y(\alpha') = y.
\]

Thus \((x, y) \in \psi \circ^{n+1} \theta\). Similarly, we can show that \( \psi \circ^{n+1} \theta \leq \theta \circ^{n+1} \psi \) and so \( \theta \) and \( \psi \) \((n+1)\)-permute. \( \square \)

A well-known consequence of Proposition 3.1 is that a distributive lattice belongs to \( \mathcal{D}_1 \) iff it is relatively complemented. It is also known that a distributive lattice has permutable congruences iff it is relatively complemented. This yields the case where \( n = 1 \) in the following:

**Theorem 3.3.** Let \( L \) be a distributive lattice. Then \( L \in \mathcal{D}_n \) iff it has \((n+1)\)-permutable congruences.

**Proof.** Suppose that \( n \geq 2 \) and \( L \in \mathcal{D}_n \). Let \( \theta, \psi \) be congruences of \( L \) and \((x, y) \in (\theta \cap \leq) \circ^{n+1} (\psi \cap \leq)\). For the sake of notational convenience, let us suppose that \( n \) is odd. Then there exist \( x_i \in L \) for \( 1 \leq i \leq n \) such that

\[
x = x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} = y
\]

and

\[
x_0 \equiv x_1(\theta), \ldots, x_n \equiv x_{n+1}(\psi).
\]

By Proposition 3.1, there are \( x_i' \in L \) for \( 1 \leq i \leq n \) such that \( x = x_0 = x_i \wedge x_i' \), \( x_i \vee x_i' = x_{i+1} \wedge x_i' \), for \( 1 \leq i \leq n \), and \( x_n \vee x_n' = x_{n+1} = y \). Let us write
\( z_0 = x \) and \( z_i = x_i \vee x'_i \) when \( 1 \leq i \leq n \). Observe that \( z_i/x_i \sim x'_i/x_i \) and \( x_{n-1} \equiv x_n(\theta) \), so that \( x \equiv x'_n(\psi) \), and furthermore, \( x_n/z_{n-1} \sim y/x_n' \) and \( z_{n-1} \equiv x_{n-1}(\theta) \), so that \( x'_n \equiv y(\theta) \). Suppose now that \( i \) is odd and \( 3 \leq i \leq n \).

Then \( x_{i-1}/z_{i-2} \sim z_{i-1}/x_{i-1} \) and \( z_{i-2} \equiv x_{i-1}(\psi) \) so that \( x_{i-1}' \equiv z_{i-1}(\psi) \), and furthermore \( z_{i}/x_{i} \sim x_{i}'/z_{i-1} \) and \( x_{i} \equiv z_{i}(\psi) \) so that \( z_{i-1} \equiv x_{i}(\psi) \). Therefore \( x_{i-1}' \equiv x_{i}(\psi) \). It follows that

\[
\begin{align*}
x \equiv x'_1(\psi), & \quad x'_2 \equiv x'_2(\psi), & \quad \ldots, & \quad x'_{n-1} \equiv x'_{n}(\psi).
\end{align*}
\]

In the event that \( i \) is even and \( 2 \leq i \leq n - 1 \), a similar argument yields \( x_{i-1}' \equiv x_{i}(\theta) \). Consequently,

\[
\begin{align*}
x'_1 \equiv x'_2(\theta), & \quad x'_3 \equiv x'_4(\theta), & \quad \ldots, & \quad x'_{n} \equiv y(\theta).
\end{align*}
\]

Thus, \((x, y) \in (\psi \land \leq) \circ^{n+1} (\theta \land \leq)\) under the assumption that \( n \) is odd. The case where \( n \) is even is dealt with similarly and so, for any \( n \geq 2 \) and arbitrary congruences \( \theta, \psi \) on \( L \), \((\theta \land \leq) \circ^{n+1} (\psi \land \leq) \subseteq (\psi \land \leq) \circ^{n+1} (\theta \land \leq)\). Therefore, by Lemma 3.2, \( L \) has \((n + 1)\)-permutable congruences.

Suppose now that \( L \) has \((n + 1)\)-permutable congruences. For \( 0 \leq i \leq n + 1 \), let \( x_i \in L \) satisfy \( x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} \). Initially, let us suppose that \( n \) is odd and so \( n = 2k + 1 \), for some \( k \geq 1 \). Define congruences \( \theta \) and \( \psi \) of \( L \) by

\[
\theta = \bigvee_{i=0}^{k} \theta(x_{2i}, x_{2i+1}) \quad \text{and} \quad \psi = \bigvee_{j=0}^{k} \theta(x_{2j+1}, x_{2j+2}).
\]

First, observe that \( \theta \land \psi = \bigvee_{i,j=0}^{k} \theta(x_{2i}, x_{2i+1}) \land \theta(x_{2j+1}, x_{2j+2}) \), since the congruence lattice of \( L \) is distributive, that \( x_{2i} \leq x_{2i+1} \leq x_{2j+1} \leq x_{2j+2} \) when \( i \leq j \), and that \( x_{2j+1} \leq x_{2j+2} \leq x_{2i} \) when \( i > j \). However, it is easy to see, using the well-known description of principal congruences of distributive lattices, that if \( a, b, c, d \in L \) and \( a \leq b \leq c \leq d \) then \( \theta(a, b) \land \theta(c, d) = \omega \).

Therefore \( \theta \land \psi = \omega \). Now, \( x_0 \equiv x_1(\theta), \ x_1 \equiv x_2(\psi), \ldots, \ x_n \equiv x_{n+1}(\psi) \) and so there exist \( x'_i \in L \), for \( 1 \leq i \leq n \), such that

\[
\begin{align*}
x_0 \leq x'_1 \leq \cdots \leq x'_n \leq x_{n+1}
\end{align*}
\]

and \( x_0 \equiv x'_1(\psi), \ x'_1 \equiv x'_2(\theta), \ldots, \ x'_n \equiv x_{n+1}(\theta) \), by Lemma 3.2. Let us define, for \( 1 \leq i \leq n \),

\[
X'_i = (x_{i-1} \lor x'_i) \land x_{i+1}
\]

and note that \( X'_i \in [x_{i-1}, x_{i+1}] \).

We claim that \( x_1 \land X'_1 = x_0 \). Indeed, \( x_1 \land X'_1 = x_1 \land (x_0 \lor x'_1 \land x_2) = (x_0 \lor x'_1) \land x_1 = x'_1 \land x_1 \) so that \( x'_1 \land X'_1 \equiv x'_1 \land x_0(\theta) = x_0 \) and \( x_1 \land X'_1 \equiv x_0 \land x_1(\psi) = x_0 \).

Therefore \( x_1 \land X'_1 \equiv x_0(\theta \land \psi) \) and so \( x_1 \land X'_1 = x_0 \).

Next we show that \( x_n \lor X'_n = x_{n+1} \). Indeed, \( x_n \lor X'_n = x_n \lor [(x_{n-1} \lor x'_n) \land x_{n+1}] = x_n \lor x'_n \) so \( x_n \lor X'_n \equiv x_{n+1} \lor x'_n(\psi) = x_{n+1} \) and \( x_n \lor X'_n \equiv x_n \lor x_{n+1}(\theta) = x_{n+1} \). Therefore \( x_n \lor X'_n \equiv x_{n+1}(\theta \land \psi) \) and so \( x_n \lor X'_n = x_{n+1} \).

It remains to show that \( x_i \lor X'_i = x_{i+1} \lor X'_{i+1} \), for \( 1 \leq i < n \). Observe that, for \( 1 \leq i < n \), \( x_i \lor X'_i = x_i \lor [(x_{i-1} \lor x'_i) \land x_{i+1}] = (x_i \lor x'_i) \land x_{i+1} \).
and suppose that \(i\) is odd. Then \(x_i \equiv x'_{i+1}(\psi)\) and \(x_i \equiv x'_{i+1}(\theta)\). Therefore, 
\[x_i \land x'_{i+1} \equiv (x_i \land x'_{i+1}) \land x_{i+1}(\theta) = x_{i+1} \land [(x_i \land x'_{i+1}) \land x_{i+2}] = x_{i+1} \land x'_{i+1};\]
that is, \(x_i \lor x'_{i+1} = x_{i+1} \land x'_{i+1}(\theta)\). Furthermore, \(x_i \lor x'_{i+1} \equiv (x_i \lor x'_{i+1}) \land x_{i+1}(\psi) = x_{i+1}\) and \(x_{i+1} \land x'_{i+1} = x_{i+1} \land [(x_i \lor x_{i+1}) \land x_{i+2}] = x_{i+1} \land (x_{i+1} \lor x'_{i+1})(\psi) = x_{i+1}\), so \(x_i \lor x'_{i+1} = x_{i+1} \land x'_{i+1}(\psi)\). Therefore, \(x_i \lor x'_{i+1} \equiv x_{i+1} \land x'_{i+1}(\theta \land \psi)\) and so \(x_i \lor x'_{i+1} = x_{i+1} \land x'_{i+1}\). When \(i\) is even and \(1 \leq i < n\) we have \(x_i \equiv x_{i+1}(\theta)\) and \(x_i' \equiv x'_{i+1}(\psi)\) and an argument similar to the one where \(i\) is odd also yields \(x_i \lor x'_{i+1} = x_{i+1} \land x'_{i+1}(\theta \land \psi)\). Thus, \(x_i \lor x'_{i+1} = x_{i+1} \land x'_{i+1}\), for any \(i\) with \(1 \leq i < n\).

In the event that \(n\) is even, so that \(n = 2k\) for some \(k \geq 1\), we define
\[
\theta = \bigvee_{i=0}^{k} \theta(x_{2i}, x_{2i+1}) \quad \text{and} \quad \psi = \bigvee_{j=1}^{k} \theta(x_{2j-1}, x_{2j}).
\]

Again we can show that \(\theta \land \psi = \omega\), proceeding in a manner similar to that for which \(n\) is odd and draw the same conclusions. In any case, \(L \in \mathcal{D}_n\) by Proposition 3.1.

**Corollary 3.4.** The length of the poset of prime ideals of a distributive lattice is \(n\) iff \(L\) has \((n+2)\)-permutable but not \((n+1)\)-permutable congruences.

**Corollary 3.5.** Every compact congruence of a distributive lattice is principal iff it has 3-permutable congruences.

**References**


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