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A TRACE FORMULA FOR TWO UNITARY OPERATORS WITH RANK ONE COMMUTATOR

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ABSTRACT. We give a short and independent proof of the Carey–Pincus trace formula for pairs of unitary operators U, V with $\operatorname{Rank}[U, V] = 1$.

Theorem. *Let U, V be two unitary operators with a one-dimensional commutator on a complex Hilbert space H. Then there exists a real measurable function $f(t, s)$ defined for t, $s \in [0, 2\pi]$ with values in the interval $[0, 1]$ such that the formula*

$$\operatorname{tr}([p(U, V), q(U, V)]) = \frac{1}{2\pi i}\int_0^{2\pi}\int_0^{2\pi}\frac{D(p, q)}{D(t, s)}(e^{it}, e^{is})f(t, s)\,dt\,ds \tag{1}$$

holds for any two polynomials p, q of the form

$$p(e^{it}, e^{is}) = \sum c_{n,n}e^{int}e^{ims} \quad \text{with } p(U, V) = \sum c_{n,m}U^nV^m,$$

and $D(p, q)/D(t, s)$ is the Jacobian of p, q with respect to t, s. The function f also has the property: $f(t, s) = 0$ if $e^{it} \notin \sigma(U)$ or $e^{is} \notin \sigma(V)$.

Comments. Trace formulas for commutators of selfadjoint operators A, B with $[A, B] \in L^1(H)$, the trace class, have been obtained by Carey and Pincus [5, 15] and Helton and Howe [12] (see also [2, 4, 7, 14, 19]). For pairs of unitary operators U, V with a trace class commutator, formula (1), with f integrable, was proved in the general setting of the type II_∞ case by Carey and Pincus [6]. Their proof is based on the introduction of a determining function for the pair U, V and applies a modification of the method presented in [5].

We introduce here a new approach based on the function

$$k(\lambda, \mu) = \frac{1}{2\pi i}\log\det(U_\lambda V_\mu U_\lambda^{-1}V_\mu^{-1}) = \frac{1}{2\pi i}\operatorname{tr}\log(U_\lambda V_\mu U_\lambda^{-1}V_\mu^{-1}),$$
$$|\lambda| < 1, \quad |\mu| < 1,$$

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where the unitary operators U_λ, V_μ are defined by

$$U_\lambda = (U - \bar{\lambda})(1 - \lambda U)^{-1} = \frac{1 - |\lambda|^2}{\lambda}(1 - \lambda U)^{-1} - \frac{1}{\lambda};$$
$$V_\mu = (V - \bar{\mu})(1 - \mu V)^{-1} = \frac{1 - |\mu|^2}{\mu}(1 - \mu V)^{-1} - \frac{1}{\mu}.$$

This is a real-valued function defined in the polydisk $D = \{\lambda, \mu \in C: |\lambda| < 1, |\mu| < 1\}$ and direct differentiation by the chain rule shows that $k(\lambda, \mu)$ is harmonic in λ and μ separately. Further properties of k can be obtained by relating it to the Krein spectral shift function, as this is done in [3]. Thus one finds $|k(\lambda, \mu)| < n$, when $\mathrm{Range}[U, V] = n < \infty$ and $k \in H^1(D)$ when $[U, V] \in L^1(H)$. In these cases k admits a representation by a double Poisson integral (see (7) below) and by differentiating this representation we come to the desired trace formula. At that, the principal function $f(t, s)$ in (1) is determined by the boundary values of k.

In order to illustrate this method we consider the case when $\mathrm{Range}[U, V] = 1$ which is very simple and can be treated directly. Although simple, this case is the most important one, (see [8, 9, 14, 16, 17, 20, 21]), has a rich theory, and provides a good insight in the nature of $f(t, s)$.

Proof of the theorem. It is easy to check that

(2)
$$[U_\lambda, V_\mu] = (1-|\lambda|^2)(1-|\mu|^2)(1-\lambda U)^{-1}(1-\mu V)^{-1}[U, V](1-\mu V)^{-1}(1-\lambda U)^{-1}.$$

We want to show that the unitary operator

$$W_{\lambda\mu} = U_\lambda V_\mu U_\lambda^{-1} V_\mu^{-1} = 1 + [U_\lambda, V_\mu] U_\lambda^{-1} V_\mu^{-1} \tag{3}$$

has, for all λ, μ, just two different eigenvalues; namely, one is 1 and the other we denote by $\alpha(\lambda, \mu)$ (cf. [6, Example (3.12), p. 76]. This is obvious: the operator $[U_\lambda, V_\mu]U_\lambda^{-1}V_\mu^{-1}$ is one-dimensional and has at most one eigenvalue $\alpha(\lambda, \mu)-1$ different from 0 and no continuous spectrum. If for some λ, μ, $W_{\lambda\mu}$ has only the eigenvalue 1, that is $\alpha(\lambda, \mu) = 1$, then $W_{\lambda\mu}$ will be the identity operator and $[U_\lambda, V_\mu] = 0$, which is impossible, as $[U, V] \neq 0$. Therefore $\alpha(\lambda, \mu) \neq 1$ for all $(\lambda, \mu) \in D$. We see that when λ, μ vary in the open unit disk, $\alpha(\lambda, \mu)$ varies continuously on the unit circle never reaching the point 1 and hence never passing through it. We can write

$$\alpha(\lambda, \mu) = \det(W_{\lambda\mu}) = \exp(2\pi i k(\lambda, \mu)), \qquad |\lambda|, |\mu| < 1, \tag{4}$$

where

$$k(\lambda, \mu) = \frac{1}{2\pi i}\log\det(W_{\lambda\mu}) = \frac{1}{2\pi i}\operatorname{tr}\log(W_{\lambda\mu}) \tag{5}$$

is a continuous function taking values in the interval $(0, 1)$. For $\log(\cdot)$ we take the branch holomorphic on $\mathbf{C}\backslash R^+$. (See also Note (a) at the end.) Differentiating $\operatorname{tr}\log(U_\lambda V_\mu U_\lambda^{-1} V_\mu^{-1})$ by the chain rule (see [10, 1.6.11, Lemma 3] or [11, IV.1.9 °]) we find that $k(\lambda, \mu)$ is harmonic in λ and μ separately (the differentiation of k is given in Note (c) at the end), that is

$$\frac{\partial^2}{\partial\lambda\partial\bar{\lambda}}k(\lambda, \mu) = \frac{\partial^2}{\partial\mu\partial\bar{\mu}}k(\lambda, \mu) = 0 \quad \text{in } D.$$

Being bounded and separately harmonic, the function $k(\lambda, \mu)$ has nontangential boundary values on the unit circles $|\lambda| = 1$, $|\mu| = 1$. In particular

$$k(e^{it}, e^{is}) = \lim_{r\to 1-}\lim_{\rho\to 1-} k(re^{it}, \rho e^{is}) \quad \text{for a.e. } t, s \in [0, 2\pi], \tag{6}$$

and the limits can be interchanged. Also, k admits the representation by a double Poisson integral

$$k(\lambda, \mu) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} P_r(\theta - t)P_\rho(\varphi - s)k(e^{it}, e^{is})\,dt\,ds, \tag{7}$$

where $\lambda = re^{i\theta}$, $\mu = \rho e^{i\varphi}$, and $P_r(\theta - t) = \dfrac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} = \operatorname{Re}\left\{\dfrac{e^{it} + \lambda}{e^{it} - \lambda}\right\}$

is the Poisson kernel for the unit disk (see [2, 13, 18]).

This formula is interesting by itself. Combined with (4) it provides a "real part" version for unitary operators of the exponential formula considered, for instance, in [8] and [17].

We proceed further by applying the differential operator $\partial^2/\partial\lambda\partial\mu$ to both sides of (7) written in the form

$$\operatorname{tr}\log(U_\lambda V_\mu U_\lambda^{-1}V_\mu^{-1}) = \frac{i}{2\pi}\iint \operatorname{Re}\left\{\frac{e^{it} + \lambda}{e^{it} - \lambda}\right\}\operatorname{Re}\left\{\frac{e^{is} + \mu}{e^{is} - \mu}\right\}k(e^{it}, e^{is})\,dt\,ds \tag{8}$$

(integration everywhere will be for t, $s \in [0, 2\pi]$ as in (7)). This gives

$$\operatorname{tr}[U(1 - \lambda U)^{-1}, V(1 - \mu V)^{-1}] = \frac{i}{2\pi}\iint \frac{e^{it}}{(e^{it} - \lambda)^2}\frac{e^{is}}{(e^{is} - \mu)^2}k(e^{it}, e^{is})\,dt\,ds. \tag{9}$$

Expanding both sides in powers of λ and μ and comparing coefficients, one finds that for all nonnegative integers n, m :

$$\begin{aligned}\operatorname{tr}[U^n, V^m] &= \frac{i}{2\pi}\iint nme^{-int}e^{-ims}k(e^{it}, e^{is})\,dt\,ds \\ &= \frac{i}{2\pi}\iint \frac{\partial}{\partial t}(e^{int})\frac{\partial}{\partial s}(e^{ims})k(e^{-it}, e^{-is})\,dt\,ds\end{aligned} \tag{10}$$

(after the substitution $t \to 2\pi - t$, $s \to 2\pi - s$).

Applying successively to (8) the operators $\partial^2/\partial\bar{\lambda}\partial\mu$, $\partial^2/\partial\lambda\partial\bar{\mu}$, $\partial^2/\partial\bar{\lambda}\partial\bar{\mu}$, and arguing in the same way, we find that (10) holds for all integers n, $m \in Z$.

Finally, a standard application of Wallach's lemma implies the trace formula (1) with $f(t, s) = k(e^{-it}, e^{-is})$ (see [5, 7, 12, 14, 19]).

If $e^{it} \notin \sigma(U)$, then the operator $(1 - e^{-it}U)^{-1}$ is bounded and when $\lambda \to e^{-it}$ and μ stays fixed, (2) implies $[U_\lambda, V_\mu] \to 0$ and hence $k(\lambda, \mu) \to 0$. The same is true when we interchange U and V, e^{it} and e^{is}, λ and μ. Therefore $f(t, s) = k(e^{-it}, e^{-is}) = 0$ when $e^{it} \notin \sigma(U)$ or $e^{is} \notin \sigma(V)$ according to (6). The proof is completed.

Some notes. (a) The operator $W_{\lambda\mu} = U_\lambda V_\mu U_\lambda^{-1} V_\mu^{-1}$, $|\lambda|, |\mu| < 1$, being unitary, has the representation

$$W_{\lambda\mu} = \int_0^{2\pi} e^{i\theta} dE_{\lambda\mu}(\theta) = \exp(iA_{\lambda\mu}),$$

$$\text{where } A_{\lambda\mu} = \int_0^{2\pi} \theta dE_{\lambda\mu}(\theta) = \frac{1}{i}\log(W_{\lambda\mu})$$

In case $[U, V]$ is one-dimensional, $\alpha(\lambda, \mu)$ is the only eigenvalue of $W_{\lambda\mu}$ different from 1 and $2\pi k(\lambda, \mu)$ is the only eigenvalue of $A_{\lambda\mu}$ different from 0 (the spectral function $E_{\lambda\mu}(\theta)$ has exactly one jump for $0 < \theta < 2\pi$). When $[U, V]$ is n-dimensional, $n > 1$, $E_{\lambda\mu}$ has (at most) n jumps in $(0, 2\pi)$ and hence $A_{\lambda\mu}$ has (at most) n nonzero eigenvalues $2\pi k_j(\lambda, \mu)$, $j = 1, 2, \dots, n$, $0 < k_j < 1$. In this case

$$k(\lambda, \mu) = \frac{1}{2\pi}\operatorname{tr}(A_{\lambda\mu}) = \sum k_j(\lambda, \mu) \in (0, n)$$

and we get (1) with $0 \leqq f \leqq n$. Also, for $\lambda = \mu = 0$ we find from (8)

$$\frac{2\pi}{i}\operatorname{tr}\log(UVU^*V^*) = \iint f(t, s)\, dt\, ds.$$

(b) We give $\alpha(\lambda, \mu)$ in explicit form. As the normal operator $[U, V]U^{-1}V^{-1} = UVU^{-1}V^{-1} - 1$ is one-dimensional, there exist a nonzero vector $\psi \in H$ and a nonzero complex number a, such that $[U, V]U^{-1}V^{-1}x = a(x, \psi)\psi$ and $VU[U, V]^*x = ([U, V]U^{-1}V^{-1})^*x = \overline{a}(x, \psi)\psi$ for all $x \in H$ (see [1, §6].

Define the vector function of norm one

$$\psi(\lambda, \mu) = (1 - \lambda U)^{-1}(1 - \mu V)^{-1}\psi / \|(1 - \lambda U)^{-1}(1 - \mu V)^{-1}\psi\|, \qquad |\lambda|, |\mu| < 1.$$

In view of (2), $\psi(\lambda, \mu)$ is a normalized eigenvector of the one-dimensional operator $[U_\lambda, V_\mu]U_\lambda^{-1}V_\mu^{-1} = U_\lambda V_\mu U_\lambda^{-1} V_\mu^{-1} - 1 = W_{\lambda\mu} - 1$, and hence of $W_{\lambda\mu}$. For the corresponding eigenvalue $\alpha(\lambda, \mu)$ we have

$$\alpha(\lambda, \mu) = \langle W_{\lambda\mu}\psi(\lambda, \mu), \psi(\lambda, \mu)\rangle,$$

and one can check directly that $\operatorname{Re}\alpha(\lambda, \mu) < 1$ in D.

(c) Differentiation of $k(\lambda, \mu)$. It is sufficient to find $\partial k(\lambda, \mu)/\partial\lambda$. Because $\partial U_\lambda/\partial\lambda = (U - \bar{\lambda})(1 - \lambda U)^{-2}U = U(1 - \lambda U)^{-1}U_\lambda$ and because $\partial U_\lambda^{-1}/\partial\lambda = -U(U - \bar{\lambda})^{-1} = -U(1 - \lambda U)^{-1}U_\lambda^{-1}$, we find

$$\frac{\partial}{\partial\lambda}2\pi i k(\lambda, \mu) = \frac{\partial}{\partial\lambda}\operatorname{tr}\log(W_{\lambda\mu}) = \operatorname{tr}\left[\left(\frac{\partial}{\partial\lambda}W_{\lambda\mu}\right)W_{\lambda\mu}^{-1}\right]$$
$$= \operatorname{tr}\{U(1-\lambda U)^{-1}U_\lambda V_\mu U_\lambda^{-1}V_\mu^{-1} - U_\lambda V_\mu U(1-\lambda U)^{-1}U_\lambda^{-1}V_\mu^{-1})V_\mu U_\lambda V_\mu^{-1}U_\lambda^{-1}\}$$
$$= \operatorname{tr}(U(1-\lambda U)^{-1} - U_\lambda V_\mu U(1-\lambda U)^{-1}V_\mu^{-1}U_\lambda^{-1})$$
$$= \operatorname{tr}\{U_\lambda(U_\lambda^{-1}U(1-\lambda U)^{-1}U_\lambda - V_\mu U(1-\lambda U)^{-1}V_\mu^{-1})U_\lambda^{-1}\}$$
$$= \operatorname{tr}(U(1-\lambda U)^{-1} - V_\mu U(1-\lambda U)^{-1}V_\mu^{-1}).$$

Obviously this does not depend on $\bar{\lambda}$, so that $\partial^2 k(\lambda, \mu)/\partial\bar{\lambda}\partial\lambda = 0$. By symmetry, $\partial^2 k(\lambda, \mu)/\partial\bar{\mu}\partial\mu = 0$, too.

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