A TRACE FORMULA FOR TWO UNITARY OPERATORS WITH RANK ONE COMMUTATOR

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Abstract. We give a short and independent proof of the Carey–Pincus trace formula for pairs of unitary operators \( U, V \) with \( \text{Rank}[U, V] = 1 \).

Theorem. Let \( U, V \) be two unitary operators with a one-dimensional commutator on a complex Hilbert space \( H \). Then there exists a real measurable function \( f(t, s) \) defined for \( t, s \in [0, 2\pi] \) with values in the interval \( [0, 1] \) such that the formula

\[
\text{tr}([p(U, V), q(U, V)]) = \frac{1}{2\pi i} \int_0^{2\pi} \int_0^{2\pi} \frac{D(p, q)}{D(t, s)} (e^{it}, e^{is}) f(t, s) \, dt \, ds
\]

holds for any two polynomials \( p, q \) of the form

\[p(e^{it}, e^{is}) = \sum c_{n,m} e^{int} e^{ims}\]

with \( p(U, V) = \sum c_{n,m} U^n V^m \), and \( D(p, q)/D(t, s) \) is the Jacobian of \( p, q \) with respect to \( t, s \). The function \( f \) also has the property: \( f(t, s) = 0 \) if \( e^{it} \notin \sigma(U) \) or \( e^{is} \notin \sigma(V) \).

Comments. Trace formulas for commutators of selfadjoint operators \( A, B \) with \([A, B] \in L^1(H)\), the trace class, have been obtained by Carey and Pincus [5, 15] and Helton and Howe [12] (see also [2, 4, 7, 14, 19]). For pairs of unitary operators \( U, V \) with a trace class commutator, formula (1), with \( f \) integrable, was proved in the general setting of the type \( \text{II}_\infty \) case by Carey and Pincus [6]. Their proof is based on the introduction of a determining function for the pair \( U, V \) and applies a modification of the method presented in [5].

We introduce here a new approach based on the function

\[k(\lambda, \mu) = \frac{1}{2\pi i} \log \det(U_{\lambda} V_{\mu} U_{\lambda}^{-1} V_{\mu}^{-1}) = \frac{1}{2\pi i} \text{tr} \log(U_{\lambda} V_{\mu} U_{\lambda}^{-1} V_{\mu}^{-1}), \quad |\lambda| < 1, \quad |\mu| < 1,\]

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where the unitary operators $U_\lambda$, $V_\mu$ are defined by
\[
U_\lambda = (U - \bar{\lambda})(1 - \lambda U)^{-1} = \frac{1 - |\lambda|^2}{\lambda}(1 - \lambda U)^{-1} - \frac{1}{\lambda};
\]
\[
V_\mu = (V - \bar{\mu})(1 - \mu V)^{-1} = \frac{1 - |\mu|^2}{\mu}(1 - \mu V)^{-1} - \frac{1}{\mu}.
\]
This is a real-valued function defined in the polydisk $D = \{\lambda, \mu \in \mathbb{C}: |\lambda| < 1, |\mu| < 1\}$ and direct differentiation by the chain rule shows that $k(\lambda, \mu)$ is harmonic in $\lambda$ and $\mu$ separately. Further properties of $k$ can be obtained by relating it to the Krein spectral shift function, as this is done in [3]. Thus one finds $|k(\lambda, \mu)| < n$, when $\text{Range}[U, V] = n < \infty$ and $k \in H^1(D)$. In these cases $k$ admits a representation by a double Poisson integral (see (7) below) and by differentiating this representation we come to the desired trace formula. At that, the principal function $f(t, s)$ in (1) is determined by the boundary values of $k$.

In order to illustrate this method we consider the case when $\text{Range}[U, V] = 1$ which is very simple and can be treated directly. Although simple, this case is the most important one, (see [8, 9, 14, 16, 17, 20, 21]), has a rich theory, and provides a good insight in the nature of $f(t, s)$.

**Proof of the theorem.** It is easy to check that
\[
[U_\lambda, V_\mu] = (1 - \lambda^2)(1 - |\mu|^2)(1 - \lambda U)^{-1}(1 - \mu V)^{-1}[U, V](1 - \mu V)^{-1}(1 - \lambda U)^{-1}.
\]
We want to show that the unitary operator
\[
W_{\lambda\mu} = U_\lambda V_\mu U_\lambda^{-1} V_\mu^{-1} = 1 + [U_\lambda, V_\mu]U_\lambda^{-1} V_\mu^{-1}
\]
has, for all $\lambda, \mu$, just two different eigenvalues; namely, one is 1 and the other we denote by $\alpha(\lambda, \mu)$ (cf. [6, Example (3.12), p. 76]). This is obvious: the operator $[U_\lambda, V_\mu]U_\lambda^{-1} V_\mu^{-1}$ is one-dimensional and has at most one eigenvalue $\alpha(\lambda, \mu) - 1$ different from 0 and no continuous spectrum. If for some $\lambda, \mu$, $W_{\lambda\mu}$ has only the eigenvalue 1, that is $\alpha(\lambda, \mu) = 1$, then $W_{\lambda\mu}$ will be the identity operator and $[U_\lambda, V_\mu] = 0$, which is impossible, as $[U, V] \neq 0$. Therefore $\alpha(\lambda, \mu) \neq 1$ for all $(\lambda, \mu) \in D$. We see that when $\lambda, \mu$ vary in the open unit disk, $\alpha(\lambda, \mu)$ varies continuously on the unit circle never reaching the point 1 and hence never passing through it. We can write
\[
\alpha(\lambda, \mu) = \det(W_{\lambda\mu}) = \exp(2\pi i k(\lambda, \mu)), \quad |\lambda|, |\mu| < 1,
\]
where
\[
k(\lambda, \mu) = \frac{1}{2\pi i} \log \det(W_{\lambda\mu}) = \frac{1}{2\pi i} \text{tr} \log(W_{\lambda\mu})
\]
is a continuous function taking values in the interval $(0, 1)$. For $\log(\cdot)$ we take the branch holomorphic on $\mathbb{C}\setminus R^+$. (See also Note (a) at the end.) Differentiating $\text{tr} \log(U_\lambda V_\mu U_\lambda^{-1} V_\mu^{-1})$ by the chain rule (see [10, 1.6.11, Lemma 3]
A TRACE FORMULA FOR TWO UNITARY OPERATORS WITH RANK ONE COMMUTATOR 159

or [11, IV. 1.9 °]) we find that \( k(\lambda, \mu) \) is harmonic in \( \lambda \) and \( \mu \) separately (the differentiation of \( k \) is given in Note (c) at the end), that is

\[
\frac{\partial^2}{\partial \lambda \partial \mu} k(\lambda, \mu) = \frac{\partial^2}{\partial \mu \partial \bar{\mu}} k(\lambda, \mu) = 0 \quad \text{in } D.
\]

Being bounded and separately harmonic, the function \( k(\lambda, \mu) \) has nontangential boundary values on the unit circles \( |\lambda| = 1, \ |\mu| = 1 \). In particular

\[
k(e^{it}, e^{is}) = \lim_{r \to 1} \lim_{\rho \to 1} k(re^{it}, \rho e^{is}) \quad \text{for a.e. } t, s \in [0, 2\pi],
\]

and the limits can be interchanged. Also, \( k \) admits the representation by a double Poisson integral

\[
k(\lambda, \mu) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} P_r(\theta - t) P_\rho(\varphi - s) k(e^{it}, e^{is}) \, dt \, ds,
\]

where \( \lambda = re^{i\theta}, \mu = \rho e^{i\varphi} \), and \( P_r(\theta - t) = \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - t)} = \text{Re} \left\{ \frac{e^{it} + \lambda}{e^{it} - \lambda} \right\} \)

is the Poisson kernel for the unit disk (see [2, 13, 18]).

This formula is interesting by itself. Combined with (4) it provides a "real part" version for unitary operators of the exponential formula considered, for instance, in [8] and [17].

We proceed further by applying the differential operator \( \frac{\partial^2}{\partial \lambda \partial \mu} \) to both sides of (7) written in the form

\[
\text{tr} \log(U, V^{-1}) = \frac{i}{2\pi} \int\int \text{Re} \left\{ \frac{e^{it} + \lambda}{e^{it} - \lambda} \right\} \text{Re} \left\{ \frac{e^{is} + \mu}{e^{is} - \mu} \right\} k(e^{it}, e^{is}) \, dt \, ds
\]

(integration everywhere will be for \( t, s \in [0, 2\pi] \) as in (7)). This gives

\[
\text{tr}[U(1 - \lambda U)^{-1}, V(1 - \mu V)^{-1}] = \frac{i}{2\pi} \int\int \frac{e^{it}}{(e^{it} - \lambda)^2} \frac{e^{is}}{(e^{is} - \mu)^2} k(e^{it}, e^{is}) \, dt \, ds.
\]

Expanding both sides in powers of \( \lambda \) and \( \mu \) and comparing coefficients, one finds that for all nonnegative integers \( n, m \):

\[
\text{tr}[U^n, V^m] = \frac{i}{2\pi} \int\int nme^{-int} e^{-ims} k(e^{it}, e^{is}) \, dt \, ds
\]

(after the substitution \( t \to 2\pi - t \), \( s \to 2\pi - s \)).

Applying successively to (8) the operators \( \frac{\partial^2}{\partial \lambda \partial \mu}, \frac{\partial^2}{\partial \lambda \partial \bar{\mu}}, \frac{\partial^2}{\partial \bar{\lambda} \partial \mu}, \frac{\partial^2}{\partial \bar{\lambda} \partial \bar{\mu}} \)

and arguing in the same way, we find that (10) holds for all integers \( n, m \in \mathbb{Z} \).

Finally, a standard application of Wallach's lemma implies the trace formula (1) with \( f(t, s) = k(e^{-it}, e^{-is}) \) (see [5, 7, 12, 14, 19]).

If \( e^{it} \notin \sigma(U) \), then the operator \( (1 - e^{-it} U)^{-1} \) is bounded and when \( \lambda \to e^{-it} \) and \( \mu \) stays fixed, (2) implies \( [U, V] \to 0 \) and hence \( k(\lambda, \mu) \to 0 \). The
same is true when we interchange $U$ and $V$, $e^{it}$ and $e^{is}$, $\lambda$ and $\mu$. Therefore
\[ f(t, s) = k(e^{-it}, e^{-is}) = 0 \] when $e^{it} \notin \sigma(U)$ or $e^{is} \notin \sigma(V)$ according to (6).
The proof is completed.

Some notes. (a) The operator $W_{\lambda \mu} = U_{\lambda} V_{\mu} U_{\lambda}^{-1} V_{\mu}^{-1}$, $|\lambda|, |\mu| < 1$, being unitary, has the representation
\[
W_{\lambda \mu} = \int_{0}^{2\pi} e^{i\theta} dE_{\lambda \mu}(\theta) = \exp(iA_{\lambda \mu}),
\]
where $A_{\lambda \mu} = \int_{0}^{2\pi} \theta dE_{\lambda \mu}(\theta) = \frac{1}{i} \log(W_{\lambda \mu})$

In case $[U, V]$ is one-dimensional, $\alpha(\lambda, \mu)$ is the only eigenvalue of $W_{\lambda \mu}$ different from 1 and $2\pi k(\lambda, \mu)$ is the only eigenvalue of $A_{\lambda \mu}$ different from 0 (the spectral function $E_{\lambda \mu}(\theta)$ has exactly one jump for $0 < \theta < 2\pi$). When $[U, V]$ is $n$-dimensional, $n > 1$, $E_{\lambda \mu}$ has (at most) $n$ jumps in $(0, 2\pi)$ and hence $A_{\lambda \mu}$ has (at most) $n$ nonzero eigenvalues $2\pi k_{j}(\lambda, \mu)$, $j = 1, 2, \ldots, n$, $0 < k_{j} < 1$. In this case
\[ k(\lambda, \mu) = \frac{1}{2\pi} \text{tr}(A_{\lambda \mu}) = \sum_{j} k_{j}(\lambda, \mu) \in (0, n) \]
and we get (1) with $0 \leq f \leq n$. Also, for $\lambda = \mu = 0$ we find from (8)
\[ \frac{2\pi}{i} \text{tr} \log(UVU^{*}V^{*}) = \iint f(t, s) dtds. \]

(b) We give $\alpha(\lambda, \mu)$ in explicit form. As the normal operator $[U, V]U^{-1}V^{-1} = UVU^{-1}V^{-1} - 1$ is one-dimensional, there exist a nonzero vector $\psi \in H$ and a nonzero complex number $a$, such that $[U, V]U^{-1}V^{-1}x = a(x, \psi)\psi$ and $VU[U, V]U^{-1}V^{-1}x = (U, V]U^{-1}V^{-1})^{*}x = \overline{a}(x, \psi)\psi$ for all $x \in H$ (see [1, §6].

Define the vector function of norm one
\[ \psi(\lambda, \mu) = (1 - \lambda U)^{-1}(1 - \mu V)^{-1}\psi / \|(1 - \lambda U)^{-1}(1 - \mu V)^{-1}\psi\|, \] $|\lambda|, |\mu| < 1$.

In view of (2), $\psi(\lambda, \mu)$ is a normalized eigenvector of the one-dimensional operator $[U_{\lambda}, V_{\mu}]U_{\lambda}^{-1}V_{\mu}^{-1} = U_{\lambda} V_{\mu} U_{\lambda}^{-1} V_{\mu}^{-1} - 1 = W_{\lambda \mu} - 1$, and hence of $W_{\lambda \mu}$. For the corresponding eigenvalue $\alpha(\lambda, \mu)$ we have
\[ \alpha(\lambda, \mu) = \langle W_{\lambda \mu} \psi(\lambda, \mu), \psi(\lambda, \mu) \rangle, \]
and one can check directly that $\text{Re} \alpha(\lambda, \mu) < 1$ in $D$. 

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(c) Differentiation of \( k(\lambda, \mu) \). It is sufficient to find \( \frac{\partial k(\lambda, \mu)}{\partial \lambda} \). Because \( \frac{\partial U_\lambda}{\partial \lambda} = (U - \lambda)(1 - \lambda U)^{-2}U = U(1 - \lambda U)^{-1}U_\lambda \) and because \( \frac{\partial U_\lambda^{-1}}{\partial \lambda} = -U(1 - \lambda U)^{-1}U_\lambda^{-1} \), we find
\[
\frac{\partial}{\partial \lambda} 2\pi i k(\lambda, \mu) = \frac{\partial}{\partial \lambda} \text{tr} \log(W_{\lambda \mu}) = \text{tr} \left[ \left( \frac{\partial}{\partial \lambda} W_{\lambda \mu} \right) W_{\lambda \mu}^{-1} \right]
\]
\[
= \text{tr} \left\{ U(1 - \lambda U)^{-1} U_\lambda V_\mu U_\lambda^{-1} V_\mu^{-1} - U_\lambda V_\mu U(1 - \lambda U)^{-1} U_\lambda^{-1} V_\mu^{-1} \right\} V_\mu U_\lambda V_\mu^{-1} U_\lambda^{-1}
\]
\[
= \text{tr} \left\{ U_\lambda^{-1} U(1 - \lambda U)^{-1} V_\mu U(1 - \lambda U)^{-1} V_\mu^{-1} \right\} U_\lambda^{-1}
\]
\[
= \text{tr} (U(1 - \lambda U)^{-1} - V_\mu U(1 - \lambda U)^{-1} V_\mu^{-1} U_\lambda^{-1}).
\]
Obviously this does not depend on \( \lambda \), so that \( \frac{\partial^2 k(\lambda, \mu)}{\partial \lambda^2} = 0 \). By symmetry, \( \frac{\partial^2 k(\lambda, \mu)}{\partial \mu^2} = 0 \), too.

References


