PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 1, September 1991

WEIGHTED NORM INEQUALITIES FOR OPERATORS OF HARDY TYPE

STEVEN BLOOM AND RON KERMAN

(Communicated by Andrew M. Bruckner)

ABSTRACT. A new proof, yielding new conditions, is given for the two-weighted norm Hardy inequality. The theorem is extended to operators with kernels behaving much like the Riemann–Liouville fractional integrals of nonnegative order.

I. HARDY'S INEQUALITY

In this paper we present a new and easily proven criterion for the Hardy operator

$$If(x) = \int_0^x f(y)\,dy, \qquad x > 0$$

to satisfy a weighted norm inequality. The criterion involves the adjoint operator

$$I^* g(x) = \int_x^\infty g(y)\,dy.$$

Using a more complicated version of the proof for I, we are able to obtain, in §II, similar criteria for weighted norm inequalities to hold for, among other operators, those of Riemann–Liouville,

$$I_\alpha f(x) = \int_0^x (x-y)^\alpha f(y)dy, \qquad x \text{ and } \alpha > 0.$$

These criteria are related to other, recently obtained, characterizations for I_α in §III.

The following special case of the result for I gives the essentials of our approach. The method of proof is similar to the classical proof of Hardy's inequality [1, p. 242–3].

Theorem 1.1. *Fix* $1 < p < \infty$. *Let* u *and* v *be nonnegative, measurable functions on* $(0, \infty)$, *with* $0 < u$, $v < \infty$ a.e. *Then,*

$$\int_0^\infty (uIf)^p \le C \int_0^\infty (vf)^p \tag{1.1}$$

Received by the editors March 9, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26D15, 26A33.
The first author's research was supported in part by a grant from Siena College.
The second author's research was supported in part by NSERC grant A4021.

for all nonnegative f, *if and only if,*

$$I^*[(v^{-1}I^*u^p)^{p'}] \le CI^*u^p < \infty, \tag{1.2}$$

where p' *is the conjugate exponent* $p/(p-1)$.

Proof. We may restrict our attention to bounded and nonnegative f that are compactly supported in $(0, \infty)$. Then an integration by parts gives the identity

$$\int_0^\infty (uIf)^p = p\int_0^\infty f(If)^{p-1}I^*u^p. \tag{1.3}$$

This simple identity is the key to our proof. For if $\int_0^\infty (vf)^p = 1$, then Hölder's inequality and another integration by parts gives

$$\begin{aligned}\int_0^\infty (uIf)^p &\le p\left[\int_0^\infty (If)^p(v^{-1}I^*u^p)^{p'}\right]^{1/p'}\\ &= p\left[\int_0^\infty I^*[(v^{-1}I^*u^p)^{p'}]d(If)^p\right]^{1/p'}.\end{aligned}$$

Now using (1.2) and a further integration by parts, we obtain

$$\int_0^\infty (uIf)^p \le C\left[\int_0^\infty I^*u^P d(If)^p\right]^{1/p'} = C\left[\int_0^\infty (uIf)^p\right]^{1/p'} < \infty,$$

and so,

$$\int_0^\infty (uIf)^p \le C.$$

To see that (1.2) is implied by (1.1), observe that for any of our f with $\int (vf)^p < \infty$, we have

$$p\int_0^\infty f(If)^{p-1}I^*u^p = \int_0^\infty (uIf)^p \le C\int_0^\infty (vf)^p < \infty.$$

This means that $I^*u^p < \infty$ on $(0, \infty)$. Substituting

$$h(\cdot) = u(\cdot)^p\chi_{(x,\infty)}(\cdot)$$

into the (equivalent) inequality

$$\int_0^\infty (v^{-1}I^*h)^{p'} \le C\int (u^{-1}h)^{p'}$$

dual to (1.1), we obtain (1.2).

II. Riemann–Liouville operators

It will be convenient to prove our result for kernels somewhat more general than those of Riemann–Liouville. We consider kernels $\varphi(x, y)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ with the following properties:

(2.1)
- (i) $\varphi(x, y) > 0$ if $x > y$;
- (ii) $\varphi(x, y)$ is nondecreasing in x and nonincreasing in y;
- (iii) $\varphi(x, y) \approx \varphi(x, z) + \varphi(z, y)$ if $y < z < x$.

Given $r \geq 0$, we set

$$T_r f(x) = \int_0^x \varphi(x, y)^r f(y)\, dy$$

and

$$T_r^* g(x) = \int_x^\infty \varphi(y, x)^r g(y)\, dy.$$

We will write T for T_1 and I for T_0.

We can now state our principal result.

Theorem 2.1. *Fix* $1 < p \leq q < \infty$. *Let* u *and* v *be nonnegative, measurable functions on* $(0, \infty)$ *with* $0 < u, v < \infty$ a.e. *Then,*

$$\left[\int_0^\infty (uTf)^q\right]^{1/q} \leq C\left[\int_0^\infty (vf)^p\right]^{1/p} \quad \textit{for all } f \geq 0 \tag{2.2}$$

if and only if

$$I^*[(v^{-1}T^*u^q)^{p'}] \leq C(I^*u^q)^{p'/q'} < \infty \tag{2.3}$$

and

$$I^*[(v^{-1}T_q^*u^q)^{p'}] \leq C(T_q^*u^q)^{p'/q'} < \infty \tag{2.4}$$

a.e. *on* $(0, \infty)$.

To prove this, we want an analog of the elementary identity (1.3) used in the proof of Theorem 1.1. This is given in the following lemma.

Lemma 2.2. *Fix* $1 < p < \infty$. *Then for all* f *and* $u \geq 0$ *with* $\int (uTf)^p < \infty$,

$$\int_0^\infty (uTf)^p \approx \int_0^\infty f(If)^{p-1}T_p^*u^p + \int_0^\infty f(Tf)^{p-1}T^*u^p. \tag{2.5}$$

Observe that when T is the Hardy operator I, so is each T_r, and both terms on the right side of (2.5) are equal to p^{-1} times the right side of (1.3).

Proof of the lemma. Given f not identically zero, there is a greatest $x_0 \geq 0$ with $f = 0$ a.e. on $[0, x_0]$, so that $If(x)$ and $Tf(x)$ are positive when $x > x_0$. Taking adjoints,

$$\int_0^\infty (uTf)^p = \int_{x_0}^\infty fT^*[wTf], \tag{2.6}$$

where $w = u^p(Tf)^{p-2}$. Again, when $x > x_0$,

$$\begin{aligned} T^*[wTf](x) &= \int_x^\infty \varphi(y, x)w(y)\, dy \left(\int_0^x + \int_x^y\right) \varphi(y, z)f(z)\, dz \\ &= \int_x^\infty \varphi(y, x)w(y)\, dy \int_0^x \varphi(y, z)f(z)\, dz \\ &\quad + \int_x^\infty f(z)\, dz \int_z^\infty \varphi(y, x)\varphi(y, z)w(y)\, dy \\ &= A(x) + B(x). \end{aligned}$$

Now, by (2.1)(iii),

$$A \approx (If)(T_2^* w) + (Tf)(T^* w),$$
$$B \approx I^*[fI_2^* w] + T^*[fT^* w]. \tag{2.7}$$

Fubini's theorem and the equivalences (2.7) and (2.6) show

$$\int_0^\infty (uTf)^p \approx \int_{x_0}^\infty f(If)T_2^* w + \int_{x_0}^\infty f(Tf)T^* w. \tag{2.8}$$

This proves the lemma when $p = 2$. Suppose then that $p > 2$. Hölder's inequality with exponents $p-1$ and $(p-1)/(p-2)$ yields

$$T_2^* w \le [T^*(u^p(Tf)^{p-1}]^{(p-2)/(p-1)}[T_p^* u^p]^{1/(p-1)}$$

and

$$T^* w \le [T^*(u^p(Tf)^{p-1}]^{(p-2)/(p-1)}[T^* u^p]^{1/(p-1)}.$$

Hence, by another use of Hölder's inequality with the same exponents,

$$\int_{x_0}^\infty f(If)T_2^* w \le \left[\int_{x_0}^\infty f(If)^{p-1}T_p^* u^p\right]^{1/(p-1)} \left[\int_{x_0}^\infty (uTf)^p\right]^{(p-2)/(p-1)}$$
$$\int_{x_0}^\infty f(If)T^* w \le \left[\int_{x_0}^\infty f(Tf)^{p-1}T^* u^p\right]^{1/(p-1)} \left[\int_{x_0}^\infty (uTf)^p\right]^{(p-2)/(p-1)}. \tag{2.9}$$

Using the estimates (2.9) and (2.8) yields, after some simplification, (2.5), but as an inequality, rather than an equivalence.

Next, suppose $1 < p < 2$. For $x > x_0$,

$$\begin{aligned} T_2^* w(x) &= \int_x^\infty \varphi(y, x)^2 u(y)^p Tf(y)^{p-2}\,dy \\ &= \int_x^\infty \varphi(y, x)^p u(y)^p [Tf(y)/\varphi(y, x)]^{p-2}\,dy \\ &\le If(x)^{p-2} T_p^* u^p(x) \end{aligned}$$

by (2.1)(ii), and

$$\begin{aligned} T^* w(x) &= \int_x^\infty \varphi(y, x) u(y)^p Tf(y)^{p-2}\,dy \\ &\le Tf(x)^{p-2} T^* u^p(x). \end{aligned}$$

Substituting these estimates in (2.8) gives one of the inequalities implicit in (2.5).

For the opposite inequality, when $p > 2$,

$$T_r^* w(x) \ge \int_x^\infty \varphi(y, x)^r u(y)^p dy \left(\int_0^x \varphi(y, z) f(z) dz\right)^{p-2}.$$

Thus, by (2.1)(ii),

$$T_2^* w(x) \ge If(x)^{p-2} T_p^* u^p(x)$$

and

$$T^* w(x) \geq Tf(x)^{p-2} T^* u^p(x).$$

These, coupled with (2.9), prove (2.5) when $p > 2$.

When $p < 2$, we appeal again to Hölder's inequality, this time with exponents $1/(p-1)$ and $1/(2-p)$.

$$\begin{aligned} T_p^* u^p(x) &= \int_x^\infty \varphi(y, x) u^p(y) \varphi(y, x)^{p-1} Tf(y)^{(p-2)(p-1)} Tf(y)^{(2-p)(p-1)}\, dy \\ &\leq [T_2^* w]^{p-1} [T^*(u^p (Tf)^{p-1})]^{2-p} \end{aligned}$$

and so, by another application of Hölder's inequality,

$$\begin{aligned} \int f(If)^{p-1} T_p^* u^p &\leq \left[\int f(If) T_2^* w\right]^{p-1} \left[\int f T^*(u^p (Tf)^{p-1})\right]^{2-p} \\ &= \left[\int f(If) T_2^* w\right]^{p-1} \left[\int (uTf)^p\right]^{2-p}. \end{aligned}$$

Similarly,

$$T^* u^p \leq [T^* w]^{p-1} [T^* u^p (Tf)^{p-1}]^{2-p}$$

and

$$\int f(Tf)^{p-1} T^* u^p \leq \left[\int f(Tf) T^* w\right]^{p-1} \left[\int (uTf)^p\right]^{2-p},$$

which, in view of the equivalence (2.8), proves (2.5).

Proof of Theorem 2.1. In proving the only if part of the theorem, we appeal to the (equivalent) inequality

$$\left[\int_0^\infty (v^{-1} T^* g)^{p'}\right]^{1/p'} \leq C \left[\int_0^\infty (u^{-1} g)^{q'}\right]^{1/q'} \tag{2.10}$$

dual to (2.2).

Suppose (2.2) holds. Given any $f > 0$ with $\int (uf)^p < \infty$, we have, by (2.2) and (2.5),

$$\int_0^\infty f(If)^{q-1} T_q^* u^q \leq C \int_0^\infty (uTf)^q \leq C \left[\int_0^\infty (vf)^q\right]^{q/p} < \infty$$

from which we conclude that $T_q^* u^q < \infty$ on $(0, \infty)$. Moreover, from (2.1)(ii),

$$T_q^* u^q(x/2) \geq \varphi(x, x/2)^q (I^* u^q)(x)$$

and so $I^* u^q$ is also finite. Now, for $r \geq 1$,

$$\begin{aligned} I^*[(v^{-1} T_r^* u^q)^{p'}](x) &= \int_x^\infty v(y)^{-p'}\, dy \left[\int_y^\infty \varphi(z, y)^r u(z)^q \chi_{(x,\infty)}(z)\, dz\right]^{p'} \\ &\leq \int_0^\infty v(y)^{-p'}\, dy \left[\int_y^\infty \varphi(z, y) \varphi(z, x)^{r-1} u(z)^q \chi_{(x,\infty)}(z)\, dz\right]^{p'} \end{aligned}$$

by (2.1)(ii),

$$\leq C\left[\int_x^\infty \varphi(z,x)^{(r-1)q'}u(z)^q dz\right]^{p'/q'}$$

by (2.10). This estimate is (2.3) when $r=1$ and gives (2.4) when $r=q$.

In the converse direction, we assume (2.3) and (2.4) and argue from (2.5) much as in the proof of Theorem 1.1. We will show that the integral $\int_0^\infty (uTf)^q$ is bounded independently of bounded, nonnegative,compactly supported f on $(0,\infty)$ with $\int (vf)^p = 1$. It is clearly finite.

Now let $(r,s)=(0,q)$ or $(1,1)$ and let $t=(s-1)q'$. Then Hölder's inequality and an integration by parts gives

$$\begin{aligned}\int_0^\infty f(T_rf)^{q-1}T_s^*u^q &\leq \left[\int_0^\infty (vf)^p\right]^{1/p}\left[\int_0^\infty (T_rf)^{p'(q-1)}(v^{-1}T_s^*u^q)^{p'}\right]^{1/p'}\\ &= \left[\int_0^\infty I^*[(v^{-1}T_s^*u^q)^{p'}]d(T_rf)^{p'(q-1)}\right]^{1/p'}\end{aligned}$$

which by assumption, is bounded by a constant times

$$\begin{aligned}J &= \left[\int_0^\infty (T_t^*u^q)^{p'/q'}d(T_rf)^{p'(q-1)}\right]^{1/p'}\\ &= \left[\int_0^\infty (I^*[-dT_t^*u^q])^{p'/q'}d(T_rf)^{p'(q-1)}\right]^{1/p'}.\end{aligned}$$

By Minkowski's inequality,

$$J \leq \left[\int_0^\infty (-dT_t^*u^q)(T_rf)^q\right]^{1/q'}$$

and another integration by parts yields

$$J \leq \begin{cases}\left[\int f(If)^{q-1}T_q^*u^q\right]^{1/q'} & \text{when } (r,s)=(0,1),\\ \left[\int (uTf)^q\right]^{1/q'} & \text{when } (r,s)=(1,1)\end{cases}$$

which, in view of (2.5), concludes the proof.

III. Other criteria

Stepanov [3] proves that the following conditions are necessary and sufficient for $T=I_\alpha$ to satisfy (2.2).

(3.1) $$(I_{\alpha p'}v^{-p'})^{1/p'}(I^*u^q)^{1/q} \leq C < \infty$$

and

(3.2) $$(Iv^{-p'})^{1/p'}(I_{\alpha q}^*u^q)^{1/q} \leq C < \infty.$$

Using different methods, Martin–Reyes and Sawyer [2] obtain similar conditions for a more general class of operators. In this section, we show that conditions (3.1) and (3.2) imply (2.3) and (2.4) respectively, with $T_r = I_{\alpha r}$; that is

$$I^*[(v^{-1}I^*_\alpha u^q)^{p'}] \le C(I^*u^q)^{p'/q'} < \infty \tag{3.3}$$

and

$$I^*[(v^{-1}I^*_{\alpha q}u^q)^{p'}] \le C(I^*_{\alpha q}u^q)^{p'/q'} < \infty. \tag{3.4}$$

The proof that (3.1) implies (3.3) is essentially contained in Stepanov's work. To see that (3.2) implies (3.4), assume first that $(I^*_{\alpha q}u^q)(0^+) = \infty$. Then $(I^*_{\alpha q}u^q)^{-p'/q} = Ig$ for some $g \ge 0$. Define v_0 by $v_0^{-p'} = v^{-p'} + g$, so $Iv_0^{-p'} \approx (I^*_{\alpha q}u^q)^{-p'/q}$. We have

$$\begin{aligned} I^*[(v^{-1}I^*_{\alpha q}u^q)^{p'}] &\le CI^*[(v_0^{-1}I^*_{\alpha q}u^q)^{p'}] \\ &\le CI^*[(v_0^{-1}Iv^{-p'})^{-q}] \\ &\le C(Iv_0^{-p'})^{1-q} \\ &\le C(I^*_{\alpha q}u^q)^{p'/q'}. \end{aligned}$$

Suppose next that $I^*_{\alpha q}u^q(0^+) < \infty$. If also $\int_0^\infty v^{-p'} < \infty$, we are done. Otherwise, define u_0 by $\int_x^\infty y^{\alpha q}u_0(y)^q\,dy = \left[\int_0^x v(y)^{-p'}\,dy\right]^{-q/p'}$ and set $u_\varepsilon = u + \varepsilon u_0$. Then u_ε and v satisfy (3.2) and $I^*_{\alpha q}u^q_\varepsilon(0^+) = \infty$, so

$$I^*[v^{-1}I^*_{\alpha q}u^q_\varepsilon)^{p'}] \le C(I^*_{\alpha q}u^q_\varepsilon)^{p'/q'} < \infty.$$

Letting $\varepsilon \to 0^+$ gives (3.4).

References

1. G. Hardy, J. Littlewood, and G. Polya, *Inequalities*, Cambridge Univ. Press, 1967.
2. F. Martin–Reyes and E. Sawyer, *Weighted norm inequalities for the Riemann–Liouville fractional integral operators*, preprint.
3. V. Stepanov, *Two-weighted estimates for Riemann–Liouville integrals*, preprint.

Department of Mathematics, Siena College, Loudonville, New York 12211

Department of Mathematics, Brock University, St. Catharines, Ontario, L2S 3A1 Canada