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COMPLETE METRICS WITH NONPOSITIVE CURVATURE ON THE DISK

DUONG MINH DUC

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ABSTRACT. We consider complete metrics on the unit disk in $\mathbb{R}^2$ that are equivalent and conformal to the Poincaré metric.

INTRODUCTION

Let $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ and $\widetilde{K}$ be a real function on D. In [6], Kazdan asked when $\widetilde{K}$ was the Gauss curvature of a complete metric conformal to the Poincaré metric on D. In [1,2,5] we have the "Yes" answer under the following condition:

(0.1) *There exist two negative constants a and b such that in a neighborhood of the boundary ∂D of D*

$$a \leq \widetilde{K} \leq b < 0.$$

We also have the "No" answer in [2] if $\widetilde{K}$ is nonnegative in a neighborhood of ∂D. In this paper we replace (0.1) by L^p-conditions weaker than the old one. It is natural to use L^p-conditions because we define the length of curves by integration.

Our method is as follows: Let K_0 be a nonpositive Hölder-continuous function on D, which is the Gauss curvature of a metric $\rho = e^{u_0}(dx^2 + dy^2)$ such that ρ is equivalent to the Poincaré metric. Consider the perturbations $\widetilde{K} = (K + K_0)$ and $v = u + u_0$ of K_0 and u_0, respectively. We try to find conditions on K such that there is a bounded solution of the equation

$$\Delta u = K_0 e^{2u_0} = (K + K_0)e^{2u_0}e^{2u} \quad \text{in } D. \tag{0.2}$$

(Note that $\Delta u_0 = -K_0 e^{2u_0}$.) When $K_0 = -1$, $e^{2u_0} = 4(1 - |x|^2)^{-2}$, and (0.2) becomes

$$\Delta u = -\frac{4}{(1 - |x|^2)^2} - \frac{4(K - 1)e^{2u}}{(1 - |x|^2)^2} \quad \text{in } D. \tag{0.2$'$}$$

Our results are the following theorems.

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Theorem 1. *Let $\widetilde{K}$ be a nonpositive Hölder-continuous function on D and K_0 a nonpositive Hölder-continuous function on D, which is the Gauss curvature of a metric conformal and equivalent to the Poincaré metric. Then*

(i) *There is a solution u in $W_0^{1,2}(D) \cap C^2(D)$ of (0.2), if*

$$\int_D \frac{|\widetilde{K} - K_0|^2}{(1 - |x|)^2}\, dx < \infty. \tag{0.3}$$

(ii) *Moreover, this solution is bounded if there are positive real numbers p and ε such that $p > 1$, $\frac{p}{p-1}(1 - \varepsilon) < 1$, and*

$$\int_D \frac{|\widetilde{K} - K_0|^{p(1+\varepsilon)}}{(1 - |x|)^{p(1+\varepsilon)^2}}\, dx < \infty. \tag{0.4}$$

Theorem 2. *Let $\widetilde{K}$ be a nonpositive Hölder-continuous function on D and K_0 a Hölder-continuous function from D into $[a, b]$, $a \le b < 0$. Then the conclusions of Theorem 1 hold.*

Remark. Here $\widetilde{K}$ may not satisfy (0.1) for any a and b. Some special examples of $\widetilde{K}$ considered in the theorems above have been given in [2].

In §§1 and 2 we prove the theorems for the case $K_0 = -1$, and we consider general cases in the last section.

1. Existence

Denote by X the family of measurable functions u such that

$$\|u\|_* = \int_D \frac{u^2}{(1 - |x|)^2}\, dx < \infty.$$

Let $r \in (0, 1)$ and $D_r = \{x \in \mathbb{R}^2 : |x| < r\}$. We have the following lemma.

Lemma 1.1. *Let $K \in X$, $r \in (0, 1)$, and $v \in W_0^{1,2}(D_r)$. Then there exists a unique solution u in $W^{2,2}(D_r)$ of the equation*

$$\Delta u = -\frac{4}{(1 - |x|^2)^2} - \frac{4(K - 1)e^{2v}}{(1 - |x|^2)^2} \quad \text{in } D_r. \tag{1.1}$$

Proof. Using the Trudinger imbedding theorem [3, 7, 9], we see that the functions in the right-hand side are in $L^2(D_r)$. Thus by [4, Theorem 8.9, p. 185] we get the lemma. □

Fix a K in X. By Lemma 1.1, the following mapping is well defined:

$$F\colon W_0^{1,2}(D_r) \mapsto W_0^{2,2}(D_r)$$
$$F(v) = \Delta^{-1}\left(-\frac{4}{(1 - |x|^2)^2} - \frac{4(K - 1)e^{2u}}{(1 - |x|^2)^2}\right).$$

We have the following lemma.

Lemma 1.2. *Let $r \in (0, 1)$ and $K \in X$. Assume $K - 1 \le 0$. Then there exists a solution v_r in $W_0^{1,2}(D_r)$ of the equation*

$$\Delta v = -\frac{4}{(1-|x|^2)^2} - \frac{4(K-1)e^{2v}}{(1-|x|^2)^2} \quad \text{in } D_r. \tag{1.2}$$

Moreover,

$$\int_{D_r} |\nabla v_r|^2\, dx \le 8C\|K\|_*^2, \tag{1.3}$$

where C is the constant in the following Hardy inequality:

$$\int_D \frac{|u|^2}{(1-|x|)^2}\, dx \le C \int_D |\nabla u|^2\, dx \quad \forall u \in W_0^{1,2}(D). \tag{1.4}$$

Proof. By [4, Lemma 9.17, p. 242] and the Sobolev imbedding theorem, we see that F is a compact continuous mapping from $W_0^{1,2}(D_r)$ into itself. Put

$$A = \{w \in W_0^{1,2}(D_r) : w = tFw \text{ for some } t \in (0, 1)\}.$$

We shall prove that A is bounded in $W_0^{1,2}(D_r)$. Indeed, let $t \in (0, 1)$ and $w \in W_0^{1,2}(D_r)$ such that

$$\Delta w = -\frac{4t}{(1-|x|^2)^2} - \frac{4t(K-1)e^{2w}}{(1-|x|^2)^2} \quad \text{in } D_r. \tag{1.5}$$

Put $w^+ = \max\{0, w\}$ and $w^- = \max\{0, -w\}$. Multiplying both sides of (1.5) by $-w$ and integrating them on D_r, we get

(1.6)
$$\begin{aligned}\int_{D_r} |\nabla w|^2\, dx &= 4t \int_{D_r} \left\{\frac{w}{(1-|x|^2)^2} + \frac{(K-1)we^{2w}}{(1-|x|^2)^2}\right\} dx, \\ &\le 4 \int_{D_r} \frac{1}{(1-|x|^2)^2}\{w^+ + (K-1)w^+e^{2w^+} - w^- - (K-1)w^-e^{-2w^-}\}\, dx.\end{aligned}$$

Since $K - 1 \le 0$, we have

$$(K-1)w^+e^{2w^+} \le (K-1)w^+ \quad \text{and} \quad -(K-1)w^-e^{-2w^-} \le -(K-1)w^-. \tag{1.7}$$

By (1.6) and (1.7), we obtain

$$\begin{aligned}\int_{D_r} |\nabla w|^2\, dx &\le 4 \int_{D_r} \frac{Kw}{(1-|x|^2)^2}\, dx \\ &\le 4C \int_{D_r} \frac{|K|^2}{(1-|x|)^2}\, dx + \frac{1}{2C} \int_{D_r} \frac{|w|^2}{(1-|x|)^2}\, dx.\end{aligned} \tag{1.8}$$

By (1.4) and (1.8), we see that

$$\int_{D_r} |\nabla u|^2\, dx \le 8C\|K\|_*^2. \tag{1.9}$$

Thus, A is bounded in $W_0^{1,2}(D_r)$. Applying the Leray-Schauder principle [10, p. 245], we get a fixed point v_r of F, which is a solution of (1.2) and satisfies (1.3). □

The proof of (i) *of Theorem* 1 *for* $K_0 = -1$. Let $\{r_m\}$ be a sequence converging to 1 in $(0, 1)$ and $u \in W_0^{1,2}(D)$ such that $\{u_m|_{D_r}\}$ weakly converges to $u|_{D_r}$ in $W_0^{1,2}(D_r)$ for any r in $(0, 1)$, where $u_m = v_{r_m}$ is defined as in Lemma 1.2. Using [3, Corollary 1.3] we see that u is a weak solution of $W_0^{1,2}(D)$ and satisfies (1.3). □

2. Boundedness

First we need the following lemma.

Lemma 2.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with piecewise-smooth boundary* $\partial\Omega$. *Let* g *be a nonnegative measurable function on* Ω *and* h *be in* $L^p(\Omega)$ *for some* $p > n/2$. *Let* v *be a nonnegative weak solution in* $W_0^{1,2}(\Omega) \cap L^\infty(\partial\Omega)$ *of the equation*

$$\Delta v = g + h \quad \text{in } \Omega.$$

Then $\|v\|_\infty$ *is finite and bounded above by a constant depending only on* p, $\|h\|_p$, *and* $\|v\|_1$.

Proof. Let k be an arbitrary positive real number and ζ an arbitrary smooth nonnegative function of compact support in Ω whose values lie between 0 and 1. Put

$$\eta(x) = \zeta^2(x)\max\{v(x) - k\,; 0\},$$
$$A_k = \{x \in \Omega : v(x) > k\}.$$

By some simple calculations, we get

$$\int_{A_k} \left\{ |\nabla v|^2\zeta^2 + \sum_{j=1}^{m} 2\frac{\partial v}{\partial x_j}(v - k)\cdot\zeta\frac{\partial \zeta}{\partial x_j} + (g + h)(v - k)\zeta^2 \right\} dx = 0.$$

Thus,

$$\int_{A_k} |\nabla v|^2\zeta^2\,dx \le \int_{A_k} \{\tfrac{1}{2}|\nabla v|^2\zeta^2 + 2|\nabla\zeta|^2(v - k)^2 + |h|(v - k)\zeta^2\}\,dx.$$

Now, following the proof of [8, Theorem 13.1, pp. 197–199], we get the lemma. □

In order to prove the boundedness of the solution u in §1, it is sufficient to show that there is a constant M such that $\|v_r\|_\infty \le M$ for every r in $(0, 1)$, where v_r is as in Lemma 1.2. Indeed, let r be in $(0, 1)$, $w_r = v_r^+$, and Ω be an open subset of $\{x \in D_r : v_r > 0\}$ with piecewise smooth boundary $\partial\Omega$. Since v_r is in $W_0^{2,2}(D_r)$, by the Sobolev imbedding theorem v_r is in $L^\infty(D_r)$. Note that w_r is a nonnegative weak solution in $W^{1,2}(\Omega)$ of

$$\Delta w_r = 4\frac{e^{2w_r} - 1}{(1 - |x|^2)^2} - \frac{4Ke^{2w_r}}{(1 - |x|^2)^2}. \tag{2.1}$$

On the other hand, we have

$$\|v_r\|_1 \le \|v_r\|_2 \le C\|\nabla u\|_2 \le 8C^2\|K\|_*, \tag{2.2}$$

and

$$\int_D \left|\frac{Ke^{2v_r}}{(1-|x|^2)^2}\right|^{1+\varepsilon} dx \le \int_D \frac{|K|^{1+\varepsilon}e^{2(1+\varepsilon)v_r}}{(1-|x|)^{(1+\varepsilon)^2+(1-\varepsilon^2)}} \le \left\{\int_D \frac{|K|^{p(1+\varepsilon)}}{(1-|x|)^{p(1+\varepsilon)^2}}\,dx\right\}^{1/p} \times \left\{\int_D \frac{e^{2p(1+\varepsilon)v_r/(p-1)}}{(1-|x|)^{p(1-\varepsilon^2)/(p-1)}}\,dx\right\}^{(p-1)/p}. \tag{2.3}$$

Using Theorem 1.3 in [3], we can estimate the last integral in (2.3) by $\|\nabla v_r\|_2$, then by $\|K\|_*$. Therefore, by applying Lemma 2.1, we see that $\{\|w_r\|_\infty\}_r$ is bounded.

Put $z_r = v_r^-$. Then z_r is a nonnegative weak solution of

$$\Delta z_r = 4\frac{1-e^{-2z_r}}{(1-|x|^2)^2} + \frac{4Ke^{-2z_r}}{(1-|x|^2)^2} \quad \text{in } \{x \in D : v_r < 0\}. \tag{2.4}$$

Arguing as above, we see that $\{\|z_r\|_\infty\}_r$ is bounded. Therefore $\{\|v_r\|_\infty\}_r$ is bounded and $\|u\|_\infty$ is finite.

3. General cases

Let K_0 be as in Theorem 1. Then there is a C^2-function u_0 on D and two positive constants such that

$$Ae^{2u_0} \le 4(1-|x|^2)^{-2} \le Be^{2u_0} \tag{3.1}$$

and

$$\Delta u_0 = -K_0 e^{2u_0}. \tag{3.2}$$

Let $K \in X$ and $v = u_0 + u$. Then we have

$$\Delta v + (K + K_0)e^{2v} = 0 \Leftrightarrow \Delta u = K_0 e^{2u_0} - (K+K_0)e^{2u_0}e^{2u}. \tag{3.3}$$

Assume that $\widetilde{K} = K + K_0 \le 0$. Then, by (3.1), (3.3) is similar to (1.1). Therefore, by a similar procedure we can prove Theorem 1 for the case $K_0 \ne -1$.

Now let $a < b < 0$ and K_0 be a C^2-function from D into $[a, b]$. By results in [1, 2], K_0 satisfies the conditions of Theorem 1. Therefore we get Theorem 2. □

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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS, TRIESTE, ITALY

Current address: Department of Mathematics, University of Hochiminh City, Hochiminh City, Vietnam