COMPLETE METRICS WITH NONPOSITIVE CURVATURE ON THE DISK

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Abstract. We consider complete metrics on the unit disk in \( \mathbb{R}^2 \) that are equivalent and conformal to the Poincaré metric.

Introduction

Let \( D = \{ x \in \mathbb{R}^2 : |x| < 1 \} \) and \( \tilde{K} \) be a real function on \( D \). In [6], Kazdan asked when \( \tilde{K} \) was the Gauss curvature of a complete metric conformal to the Poincaré metric on \( D \). In [1,2,5] we have the "Yes" answer under the following condition:

(0.1) There exist two negative constants \( a \) and \( b \) such that in a neighborhood of the boundary \( \partial D \) of \( D \)

\[ a < \tilde{K} < b < 0. \]

We also have the "No" answer in [2] if \( \tilde{K} \) is nonnegative in a neighborhood of \( \partial D \). In this paper we replace (0.1) by \( L^p \)-conditions weaker than the old one. It is natural to use \( L^p \)-conditions because we define the length of curves by integration.

Our method is as follows: Let \( K_0 \) be a nonpositive Hölder-continuous function on \( D \), which is the Gauss curvature of a metric \( \rho = e^{u_0}(dx^2 + dy^2) \) such that \( \rho \) is equivalent to the Poincaré metric. Consider the perturbations \( \tilde{K} = (K + K_0) \) and \( \tilde{v} = v + u_0 \) of \( K_0 \) and \( u_0 \), respectively. We try to find conditions on \( K \) such that there is a bounded solution of the equation

(0.2) \[ \Delta u = K_0 e^{2u_0} = (K + K_0)e^{2u_0}e^{2u} \quad \text{in } D. \]

(Note that \( \Delta u_0 = -K_0 e^{2u_0} \).) When \( K_0 = -1 \), \( e^{2u_0} = 4(1 - |x|^2)^{-2} \), and (0.2) becomes

(0.2') \[ \Delta u = -\frac{4}{(1 - |x|^2)^2} - \frac{4(K - 1)e^{2u}}{(1 - |x|^2)^2} \quad \text{in } D. \]

Our results are the following theorems.

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Theorem 1. Let \( \tilde{K} \) be a nonpositive Hölder-continuous function on \( D \) and \( K_0 \) a nonpositive Hölder-continuous function on \( D \), which is the Gauss curvature of a metric conformal and equivalent to the Poincaré metric. Then

(i) There is a solution \( u \) in \( W^{1,2}_0(D) \cap C^2(D) \) of (0.2), if

\[
\int_D \frac{|\tilde{K} - K_0|^2}{(1 - |x|)^2} \, dx < \infty.
\]  

(ii) Moreover, this solution is bounded if there are positive real numbers \( p \) and \( \varepsilon \) such that \( p > 1 \), \( \frac{p}{p-1}(1 - \varepsilon) < 1 \), and

\[
\int_D \frac{|\tilde{K} - K_0|^{p(1+\varepsilon)}}{(1 - |x|)^{p(1+\varepsilon)^2}} \, dx < \infty.
\]

Theorem 2. Let \( \tilde{K} \) be a nonpositive Hölder-continuous function on \( D \) and \( K_0 \) a Hölder-continuous function from \( D \) into \([a, b]\), \( a < b < 0 \). Then the conclusions of Theorem 1 hold.

Remark. Here \( \tilde{K} \) may not satisfy (0.1) for any \( a \) and \( b \). Some special examples of \( \tilde{K} \) considered in the theorems above have been given in [2].

In §§1 and 2 we prove the theorems for the case \( K_0 = -1 \), and we consider general cases in the last section.

1. Existence

Denote by \( X \) the family of measurable functions \( u \) such that

\[
\|u\|_* = \int_D \frac{u^2}{(1 - |x|)^2} \, dx < \infty.
\]

Let \( r \in (0, 1) \) and \( D_r = \{ x \in \mathbb{R}^2 : |x| < r \} \). We have the following lemma.

Lemma 1.1. Let \( K \in X \), \( r \in (0, 1) \), and \( v \in W^{1,2}_0(D_r) \). Then there exists a unique solution \( u \) in \( W^{2,2}_0(D_r) \) of the equation

\[
\Delta u = -\frac{4}{(1 - |x|^2)^2} - \frac{4(K - 1)e^{2v}}{(1 - |x|^2)^2} \quad \text{in } D_r.
\]

Proof. Using the Trudinger imbedding theorem [3, 7, 9], we see that the functions in the right-hand side are in \( L^2(D_r) \). Thus by [4, Theorem 8.9, p. 185] we get the lemma. \( \square \)

Fix a \( K \) in \( X \). By Lemma 1.1, the following mapping is well defined:

\[
F : W^{1,2}_0(D_r) \rightarrow W^{2,2}_0(D_r)
\]

\[
F(v) = \Delta^{-1} \left( -\frac{4}{(1 - |x|^2)^2} - \frac{4(K - 1)e^{2v}}{(1 - |x|^2)^2} \right).
\]

We have the following lemma.
Lemma 1.2. Let \( r \in (0, 1) \) and \( K \in X \). Assume \( K - 1 \leq 0 \). Then there exists a solution \( v_r \) in \( W_{0}^{1,2}(D_r) \) of the equation

\[
\Delta v = -\frac{4}{(1 - |x|^2)^2} - \frac{4(K - 1)e^{2v}}{(1 - |x|^2)^2} \quad \text{in } D_r.
\]

Moreover,

\[
\int_{D_r} |\nabla v_r|^2 \, dx \leq 8C|K|^2,
\]

where \( C \) is the constant in the following Hardy inequality:

\[
\int_{D} \frac{|u|^2}{(1 - |x|^2)^2} \, dx \leq C \int_{D} |\nabla u|^2 \, dx \quad \forall u \in W_{0}^{1,2}(D).
\]

Proof. By [4, Lemma 9.17, p. 242] and the Sobolev imbedding theorem, we see that \( F \) is a compact continuous mapping from \( W_{0}^{1,2}(D_r) \) into itself. Put

\[
A = \{ w \in W_{0}^{1,2}(D_r) : w = tFw \text{ for some } t \in (0, 1) \}.
\]

We shall prove that \( A \) is bounded in \( W_{0}^{1,2}(D_r) \). Indeed, let \( t \in (0, 1) \) and \( w \in W_{0}^{1,2}(D_r) \) such that

\[
\Delta w = -\frac{4t}{(1 - |x|^2)^2} - \frac{4t(K - 1)e^{2w}}{(1 - |x|^2)^2} \quad \text{in } D_r.
\]

Put \( w^+ = \max\{0, w\} \) and \( w^- = \max\{0, -w\} \). Multiplying both sides of (1.5) by \(-w\) and integrating them on \( D_r \), we get

\[
\int_{D_r} |\nabla w|^2 \, dx = 4t \int_{D_r} \left\{ \frac{w}{(1 - |x|^2)^2} + \frac{(K - 1)we^{2w}}{(1 - |x|^2)^2} \right\} \, dx,
\]

\[
\leq 4 \int_{D_r} \frac{1}{(1 - |x|^2)^2} \{ w^+ + (K - 1)we^{2w^+} - w^- - (K - 1)we^{-2w^-}\} \, dx.
\]

Since \( K - 1 \leq 0 \), we have

\[
(K - 1)we^{2w^+} \leq (K - 1)w^+ \quad \text{and} \quad -(K - 1)we^{-2w^-} \leq -(K - 1)w^-.
\]

By (1.6) and (1.7), we obtain

\[
\int_{D_r} |\nabla w|^2 \, dx \leq 4 \int_{D_r} \frac{Kw}{(1 - |x|^2)^2} \, dx
\]

\[
\leq 4C \int_{D_r} \frac{|K|^2}{(1 - |x|)^2} \, dx + \frac{1}{2C} \int_{D_r} \frac{|w|^2}{(1 - |x|)^2} \, dx.
\]

By (1.4) and (1.8), we see that

\[
\int_{D_r} |\nabla w|^2 \, dx \leq 8C|K|^2.
\]
Thus, $A$ is bounded in $W^{1,2}_0(D_r)$. Applying the Leray-Schauder principle [10, p. 245], we get a fixed point $v_r$ of $F$, which is a solution of (1.2) and satisfies (1.3). □

The proof of (i) of Theorem 1 for $K_0 = -1$. Let $\{r_m\}$ be a sequence converging to 1 in $(0, 1)$ and $u \in W^{1,2}_0(D)$ such that $\{u_m|_{D_r}\}$ weakly converges to $u|_{D_r}$ in $W^{1,2}_0(D_r)$ for any $r$ in $(0, 1)$, where $u_m = v_{r_m}$ is defined as in Lemma 1.2. Using [3, Corollary 1.3] we see that $u$ is a weak solution of $W^{1,2}_0(D)$ and satisfies (1.3). □

2. BOUNDEDNESS

First we need the following lemma.

Lemma 2.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with piecewise-smooth boundary $\partial\Omega$. Let $g$ be a nonnegative measurable function on $\Omega$ and $h$ be in $L^p(\Omega)$ for some $p > n/2$. Let $v$ be a nonnegative weak solution in $W^{1,2}_0(\Omega) \cap L^\infty(\partial\Omega)$ of the equation

$$\Delta v = g + h \quad \text{in} \quad \Omega.$$ 

Then $\|v\|_\infty$ is finite and bounded above by a constant depending only on $p$, $\|h\|_p$, and $\|v\|_1$.

Proof. Let $k$ be an arbitrary positive real number and $\zeta$ an arbitrary smooth nonnegative function of compact support in $\Omega$ whose values lie between 0 and 1. Put

$$\eta(x) = \zeta^2(x) \max\{v(x) - k ; 0\},$$

$$A_k = \{x \in \Omega : v(x) > k\}.$$ 

By some simple calculations, we get

$$\int_{A_k} \left\{ |\nabla v|^2 \zeta^2 + \sum_{j=1}^m \frac{\partial v}{\partial x_j} (v - k) \cdot \zeta \frac{\partial \zeta}{\partial x_j} + (g + h)(v - k) \zeta^2 \right\} \, dx = 0.$$ 

Thus,

$$\int_{A_k} |\nabla v|^2 \zeta^2 \, dx \leq \int_{A_k} \left\{ \frac{1}{2} |\nabla v|^2 \zeta^2 + 2|\nabla \zeta|^2 (v - k)^2 + |h|(v - k) \zeta^2 \right\} \, dx.$$ 

Now, following the proof of [8, Theorem 13.1, pp. 197–199], we get the lemma. □

In order to prove the boundedness of the solution $u$ in §1, it is sufficient to show that there is a constant $M$ such that $\|u_r\|_\infty \leq M$ for every $r$ in $(0, 1)$, where $v_r$ is as in Lemma 1.2. Indeed, let $r$ be in $(0, 1)$, $w_r = v_r^+$, and $\Omega$ be an open subset of $\{x \in D_r : v_r > 0\}$ with piecewise smooth boundary $\partial\Omega$. Since $v_r$ is in $W^{2,2}_0(D_r)$, by the Sobolev imbedding theorem $v_r$ is in $L^\infty(D_r)$. Note that $w_r$ is a nonnegative weak solution in $W^{1,2}(\Omega)$ of

$$\Delta w_r = 4 \frac{e^{2w_r} - 1}{(1 - |x|^2)^2} - \frac{4Ke^{2w}}{(1 - |x|^2)^2}.$$
On the other hand, we have
\[
\|v_r\|_1 \leq \|v_r\|_2 \leq C\|\nabla u\|_2 \leq 8C^2\|K\|_*,
\]
and
\[
\int_D \left| \frac{Ke^{2v_r}}{(1 - |x|^2)^2} \right|^{1+\epsilon} dx \leq \int_D \frac{|K|^{1+\epsilon} e^{2(1+\epsilon)v_r}}{(1 - |x|)^{(1+\epsilon)^2 + (1-\epsilon^2)}} dx \leq \left\{ \int_D \frac{|K|^{p(1+\epsilon)}}{(1 - |x|)^{p(1+\epsilon)^2}} dx \right\}^{1/p} \times \left\{ \int_D \frac{e^{2p(1+\epsilon)v_r/(p-1)}}{(1 - |x|)^{p(1-\epsilon^2)/(p-1)}} dx \right\}^{(p-1)/p}.
\]

Using Theorem 1.3 in [3], we can estimate the last integral in (2.3) by \(\|\nabla v_r\|_2\), then by \(\|K\|_*\). Therefore, by applying Lemma 2.1, we see that \(\{\|w_r\|_\infty\}_r\) is bounded.

Put \(z_r = v_r^-\). Then \(z_r\) is a nonnegative weak solution of
\[
\Delta z_r = 4 \frac{1 - e^{-2z_r}}{(1 - |x|^2)^2} + \frac{4Ke^{-2z_r}}{(1 - |x|^2)^2} \quad \text{in} \quad \{x \in D : v_r < 0\}.
\]

Arguing as above, we see that \(\{\|z_r\|_\infty\}_r\) is bounded. Therefore \(\{\|v_r\|_\infty\}_r\) is bounded and \(\|u\|_\infty\) is finite.

3. General cases

Let \(K_0\) be as in Theorem 1. Then there is a \(C^2\)-function \(u_0\) on \(D\) and two positive constants such that
\[
Ae^{2u_0} \leq 4(1 - |x|^2)^{-2} \leq Be^{2u_0}
\]
and
\[
\Delta u_0 = -Ke^{2u_0}.
\]

Let \(K \in X\) and \(v = u_0 + u\). Then we have
\[
\Delta v + (K + K_0)e^{2v} = 0 \Rightarrow \Delta u = K_0e^{2u_0} - (K + K_0)e^{2u_0}e^{2u}.
\]
Assume that \(\tilde{K} = K + K_0 \leq 0\). Then, by (3.1), (3.3) is similar to (1.1). Therefore, by a similar procedure we can prove Theorem 1 for the case \(K_0 \neq -1\).

Now let \(a < b < 0\) and \(K_0\) be a \(C^2\)-function from \(D\) into \([a, b]\). By results in [1, 2], \(K_0\) satisfies the conditions of Theorem 1. Therefore we get Theorem 2. □

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